Derivation of Appleton's equation Aug 2010

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This is a derivation of Appleton's equation, which is the equation for the index of refraction of a cold plasma for whistler waves. The index of refraction of a wave is a measure of how the wave slows down when it travels in a medium. If the wave moves in 1D its magnitude is $n = \frac{c}{\mathrm{v}_{phase}}$ where c is the speed of light in vacuum.

To do this we need to introduce some vector calculus.

1) The curl of a vector $\nabla \times \vec{E} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix}$

2) The divergence of a vector $\nabla \bullet \vec{E} = \frac{\partial E_x}{\partial x}\hat{i} + \frac{\partial E_y}{\partial y}\hat{j} + \frac{\partial E_z}{\partial z}\hat{k}$

The physical meaning of these were discussed in the lecture and will not be repeated other than stating that a vector field (for example velocity) has a curl then, there is rotation in the field. For a velocity field this means there is a tornado if the velocity is that of air, or a whirlpool if it is the velocity in water. If the divergence of a vector is zero that means for any given volume (lets consider the velocity of air) that as much air enters the volume as leaves. If it is not zero there is a source or sink of air in the volume (for example a vacuum cleaner nozzle.)

To find the index of refraction we have to consider the force law for a plasma and since it is a wave we also consider the equations for the electric and magnetic fields. The force equation is by definition $\vec{F} = m\vec{a}$ (a is the acceleration). For the case of gravity $\vec{a} = -g\hat{j}$ g is a constant and gravity points downward near the earth. For an element of mass m, with charge q the force equation is more complicated

3) $m(\left(\frac{\partial \vec{\mathrm{v}}}{\partial t}\right) + (\vec{\mathrm{v}} \bullet \nabla)\vec{\mathrm{v}}) = -\nabla p + q(\vec{E} + \vec{\mathrm{v}} \times \vec{B}) - m\nu\vec{\mathrm{v}}$

There are two equations one for the ions and one for the electrons. We now start making assumptions

Assumption I The wave is a high frequency wave and the ions being more massive than the electrons do not have any time to move when the wave passes by. In this limit we can think of the ions to be a sort of crystal. They are equally spaced in three dimensions and fill all of space. We therefore only have one force equations where m is the electron mass.

What are the terms in equation 3? The second term $m(\vec{\mathrm{v}} \bullet \nabla)\vec{\mathrm{v}})$ is the part of the force due to spatial changes in the velocity field. Note that the symbol ∇ is called the gradient and is defined in rectangular coordinates by

4) $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

This term is much smaller than all the others so we will ignore it. Equation 3 now becomes

4) $m\left(\frac{\partial \vec{\mathrm{v}}}{\partial t}\right) = -\nabla p + q(\vec{E} + \vec{\mathrm{v}} \times \vec{B}) - m\nu\vec{\mathrm{v}}$

The other terms are the pressure gradient ∇p, the electromagnetic forces, and the last term on the right hand sidse is the force experienced by a particle because it is colliding with other particles. If you are strolling down the street and are hit by a fast basketball you will feel a force and can get shoved to the side. ν is the collision frequency. If there are no collisions this term is zero. Also the faster a particle is moving the more collisions per second it will experience. This gives the velocity dependence.

Assumption II. We will next assume that the background electrons are ice cold and will only bob back and forth when the wave passes by. In the LAPTAG experiment the whistlers are launched in the cold afterglow plasma. It has a temperature of about 0.2 eV or 2400 degrees Kelvin. This would be more than uncomfortably hot if you lived in it (you would fry!) but for a plasma it is considered cold. If the particles are ice cold, there wont be a difference in pressure from place to place and $\nabla p = 0$. Now equation 4 becomes:

5) $m\left(\frac{\partial \vec{\mathrm{v}}}{\partial t}\right) = +q(\vec{E} + \vec{\mathrm{v}} \times \vec{B}) - m\nu\vec{\mathrm{v}}$

This is as simple as we can make it.
The next equations we have to use are the equations of electricity and magnetism. They are called Maxwell's equations.

6) $\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$. This is called Faraday's law and relates the change of magnetic field in time to the curl (rotation) of the electric field.

7) $\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{j}} + \mu_0 \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$ This is Amperes law as modified by Maxwell. It relates changing electric fields and currents to the curl (rotation) of the magnetic field. Here $\vec{J}$ is the current density in the plasma due to currents of the whistler wave. There are two more Maxwell's equations but use of them in this derivation is not necessary, so we wont discuss them.

These are all differential equations, they involve derivatives in space and time. In general these are hard to solve but we will use a trick. It is called Fourier analysis.

We have reviewed this in the past and a short introduction is left as an appendix at the end of this discussion.
Finally, the Fourier trick we will use is to assume that we have a wave solution and that a particular whistler wave at a given frequency has a strength given by the A or B coefficient in the Fourier series (if it's a sine or a cosine respectively). That is each whistler wave will be of the form:

8) $\vec{B}_1(\vec{r},t) = Be^{i(\vec{\mathbf{k}}\bullet\vec{r}-\omega t)}$ where B is the magnetic field of the wave. Here we note that the wave is also a function of space and instead of a one dimensional solution, the wave can spread in all three directions.
$\vec{\mathbf{k}}\cdot\vec{r} = k_x x + k_y y + k_z z$. Also, the sines and cosines are connected to the exponential solutions because of Euler's theorem

9) $e^{i\theta} = \cos\theta + i\sin\theta$, which for our waves

9') $e^{i(\vec{\mathbf{k}}\bullet\vec{r}-\omega t)} = \cos(\vec{\mathbf{k}}\bullet\vec{r}-\omega t) + i\sin(\vec{\mathbf{k}}\bullet\vec{r}-\omega t)$.
If we take the real part of this we get the cosine, the imaginary part gives us the sine.

The purpose of going to Fourier land was to convert the partial differential equations (PDE's) to algebraic ones. You will see that the algebra is bad enough but not as bad as solving the PDE's. Consider Faraday's law , $\nabla\times\vec{\mathbf{E}} == -\frac{\partial\vec{\mathbf{B}}}{\partial t}$, equation 6 and Fourier solutions such as Equation 8. Taking the time derivative $\frac{\partial\vec{B}}{\partial t}$
$\frac{\partial}{\partial t}\vec{B}_1(\vec{r},t) = -i\omega Be^{i(\vec{\mathbf{k}}\bullet\vec{r}-\omega t)} = -i\omega\vec{B}_1(\vec{r},t)$. The left hand side was done in lecture but using the definition of the curl equation 6 becomes

(10) $i\vec{\mathbf{k}}\times\vec{\mathbf{E}} = i\omega\vec{B}$ $\vec{\mathbf{k}}\times\vec{\mathbf{E}} = \omega\vec{B}$
Here k is the wavenumber and its magnitude is $|\vec{\mathbf{k}}| = k = \frac{2\pi}{\lambda}$. The vector $\vec{\mathbf{k}}$ points in the direction of the phase velocity. This is the direction that a maxima or a minima of the wave moves.
If we then use the generals Ampere law $\nabla\times\vec{\mathbf{B}} == \mu_0\vec{\mathbf{j}} + \mu_0\varepsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t}$ we get (try it!)

11) $i\vec{\mathbf{k}}\times\vec{\mathbf{B}} == \mu_0(-en\vec{\mathrm{v}}) - i\omega\mu_0\varepsilon_0\vec{E}$

We now how to revisit the force equation (5) $m(\left(\frac{\partial\vec{\mathrm{v}}}{\partial t}\right) = +q(\vec{E} + \vec{\mathrm{v}}\times\vec{B}) - m\nu\vec{\mathrm{v}}$.

B is the total magnetic field which consists of the magnetic field of the wave as well as the background magnetic field in the plasma. $B(\vec{r},t)=\vec{B}_0+\vec{B}_w(\vec{r},t) \quad \vec{B}_w << \vec{B}_0$. The field of the wave is B_w and is must smaller that the background field. We want to solve for the wave field but in equation (5) we can make the very good assumption that

12) $m\left(\frac{\partial \vec{v}}{\partial t}\right)=+q(\vec{E}+\vec{v}\times\vec{B}_0)-m\nu\vec{v}$

Note that **v** is the velocity of the particles (electrons) in the field of the wave so we cant drop it and **E** is the electric field of the wave. Using our Fourier analysis equation 12 becomes:

13) $-i\omega m\vec{v}=-e\left(\vec{E}+\vec{v}\times\left(\vec{B}_0\right)\right)-m\nu\vec{v}$

Equations 9,11,and 13 are the algebraic equations we must solve. Note that these are vector equations so there are really 9 equations here. We have to solve for 3 velocities of the particles , 3 E fields of the wave and 3 B fields of the wave so we have 9 equations with 9 unknowns. We can rewrite (9) as

14) $\frac{1}{\omega}\vec{\mathbf{k}}\times\vec{\mathbf{E}}=\vec{B}_w=\vec{B}$

Instead of carry the w subscript we assume that **E,B,v** are all due to the wave and B_0 is the background field
Next we substitute B from equation 14) into equation 11)

15) $i\vec{\mathbf{k}}\times\left(\frac{1}{\omega}\vec{\mathbf{k}}\times\vec{\mathbf{E}}\right)=-\mu_0\left(en\vec{v}\right)-i\omega\mu_0\varepsilon_0\vec{E}$

We can simplify the right hand side which, has a double cross product using a vector identity. For any three vectors A,B,C $\quad \vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\bullet\vec{C}\right)-\vec{C}\left(\vec{A}\bullet\vec{B}\right)$. Using this in equation 15: $\vec{\mathbf{k}}\times\left(\vec{\mathbf{k}}\times\vec{E}\right)=\vec{\mathbf{k}}\left(\vec{\mathbf{k}}\bullet\vec{E}\right)-k^2\vec{E}$ and now the equation is

16) $i\vec{\mathbf{k}}\left(\vec{\mathbf{k}}\bullet\vec{E}\right)-ik^2\vec{E}=-\mu_0\left(\omega en\vec{v}\right)-i\omega^2\mu_0\varepsilon_0\vec{E}$

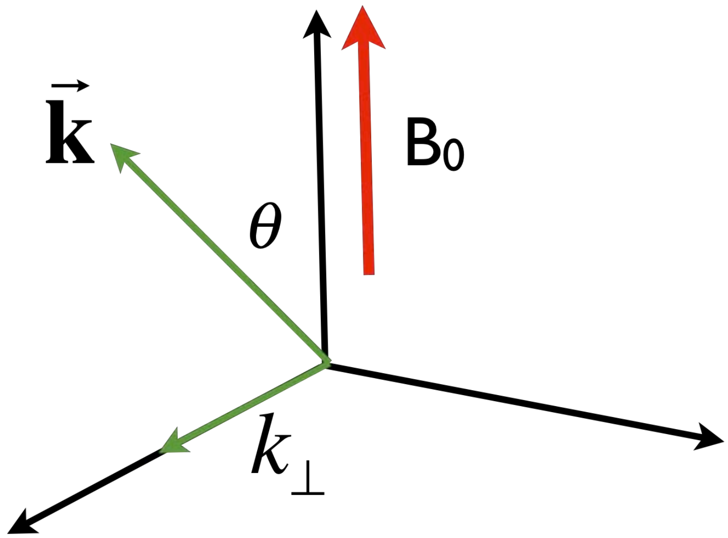
From the theory of electricity and magnetism by Maxwell $c^2=\frac{1}{\mu_0\varepsilon_0}$ the speed of light is related to the fundamental constants of electricity and magnetism. (Great triumph !)

17) $\left(\frac{i\omega e n_0\vec{v}}{\varepsilon_0 c^2}\right)=\frac{\omega^2}{c^2}\vec{E}+\vec{\mathbf{k}}\left(\vec{\mathbf{k}}\bullet\vec{E}\right)-k^2\vec{E}$

Finally we the force equation $-\omega m\vec{v}=ie\left(\vec{E}+\vec{v}\times\left(\vec{B}_0\right)\right)+im\nu\vec{v}$ can be manipulated with a bit of algebra to get:

18) $\left(1+\frac{i\nu}{\omega}\right)\vec{v}+\frac{ie}{m\omega}\left(\vec{v}\times\vec{B}_0\right)=-\frac{ie}{m\omega}\vec{E}$

Next lets do the problem in the simplest coordinate system possible. Let the background magnetic field Bo be in the z direction. The wave will propagate at some angle with respect to the background field. Call the angle theta and choose the x-y-z axis such that θ is in the x-z plane



Therefore in this system: $\vec{B}_0 = B_0\hat{k}$; $\vec{\mathbf{k}} = k\sin\theta\hat{i} + k\cos\theta\hat{k}$

Let the algebra begin!

First lets tackle the left hand side of equation 17

$$\vec{\mathbf{k}}\left(\vec{\mathbf{k}}\bullet\vec{E}\right)-k^2\vec{E}=(k\sin\theta\hat{i}+k\cos\theta\hat{k})\{\left(k\sin\theta\hat{i}+k\cos\theta\hat{k}\right)\bullet\left(E_x\hat{i}+E_y\hat{j}+E_z\hat{k}\right)$$
$$-k^2\left(E_x\hat{i}+E_y\hat{j}+E_z\hat{k}\right)$$

then

$$\vec{\mathbf{k}}\left(\vec{\mathbf{k}}\bullet\vec{E}\right)-k^2\vec{E}=(k^2\sin^2\theta E_x\hat{i}+k^2\sin\theta\cos\theta E_z\hat{i}-k^2E_x\hat{i})$$
$$-k^2E_{y\hat{j}}$$
$$\text{-k}^2(E_z+E_x\cos\theta\sin\theta+E_z\cos^2\theta)\hat{k}$$

So this has three components as shown above. Now we must equate this to the components of the right hands side

For the x component:

$$\left(\frac{i\omega e n_0 \mathrm{v}_x}{\varepsilon_0 c^2}\right) = \frac{\omega^2}{c^2} E_x + k^2 \cos\theta\left(\sin\theta E_z - \cos\theta E_x\right)$$

$$\left(\frac{i e n_0 \mathrm{v}_x}{\varepsilon_0 \omega}\right) = E_x + \frac{k^2 c^2}{\omega^2}\cos\theta\left(\sin\theta E_z - \cos\theta E_x\right)$$

$$\left(\frac{i e n_0 \mathrm{v}_x}{\varepsilon_0 \omega}\right) = E_x\left(1 - \frac{k^2 c^2}{\omega^2}\cos^2\theta\right) + \frac{k^2 c^2}{\omega^2}\cos\theta\sin\theta E_z$$

For the y component:

$$\left(\frac{i\omega e n_0 \mathrm{v}_y}{\varepsilon_0 c^2}\right) = \frac{\omega^2}{c^2} E_y + \text{-}\mathrm{k}^2 E_y$$

$$\left(1 - \frac{k^2 c^2}{\omega^2}\right) E_y = \left(\frac{i e n_0 \mathrm{v}_y}{\varepsilon_0 \omega}\right)$$

and the z component:

$$\left(\frac{i\omega e n_0 \mathrm{v}_z}{\varepsilon_0 c^2}\right) = \frac{\omega^2}{c^2} E_z + \mathrm{k}^2\left(\sin\theta E_x - \sin\theta E_z\right)$$

$$\left(\frac{i e n_0 \mathrm{v}_z}{\varepsilon_0 \omega}\right) = E_z + \frac{\mathrm{k}^2 c^2}{\omega^2}\left(\sin\theta E_x - \sin\theta E_z\right)$$

The index of refraction (remember that's what we are looking for!) is $\eta = \frac{kc}{\omega}$. We can the write these equations all at once using matrix form

$$\text{(19)} \quad \begin{pmatrix} \left(1 - \eta^2\cos^2\theta\right) & 0 & \eta^2\sin\theta\cos\theta \\ 0 & \left(1 - \eta^2\right) & 0 \\ \eta^2\sin\theta\cos\theta & 0 & \left(1 - \eta^2\sin^2\theta\right) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \left(\frac{ien}{\varepsilon_0 \omega}\right) \begin{pmatrix} \mathrm{v}_x \\ \mathrm{v}_y \\ \mathrm{v}_z \end{pmatrix}$$

Note that this is a matrix. If it is written for any two vectors (E,B) as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

The e component of the operation is $D_x = a_{11}E_x + a_{12}E_y + a_{13}E_z$. This can be extended into 4 or more dimensions and written more generally as $D_\alpha = \sum_{\beta=\alpha}^{\gamma} a_{\alpha\beta}B_\beta$. If it's a 4D world then $\gamma = 4$.

How to solve this? Equation (19) relates **E** and v. If we can get another equation relating **E** and **v** we can eliminate **v**. The equation that will do this is the force equation (18) $-i\omega m\vec{\mathrm{v}} = -e\left(\vec{E} + \vec{\mathrm{v}} \times \left(\vec{B}_0\right)\right) - m\nu\vec{\mathrm{v}}$.

We automatically get this into determinant form by doing the cross product.

$$\vec{\mathrm{v}} \times \vec{B} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \mathrm{v}_x & \mathrm{v}_y & \mathrm{v}_z \\ 0 & 0 & B_0 \end{pmatrix} = B_0\left(\mathrm{v}_y\hat{i} - \mathrm{v}_x\hat{j}\right)$$

The x component is:

$$-i\omega m\mathrm{v}_x = -e\left(\vec{E} + \mathrm{v}_y B_0\right) - m\nu\mathrm{v}_x$$

mult by i: $\omega m\mathrm{v}_x = -ie\left(\vec{E} + \mathrm{v}_y B_0\right) - im\nu\mathrm{v}_x$ divide by mω

$$\left(1 + \frac{i\nu}{\omega}\right)\mathrm{v}_x + i\frac{eB}{m}\mathrm{v}_y = -\frac{ie}{m\omega}E_x$$

For y and z:

$$-i\frac{\omega_{ce}}{\omega}\mathrm{v}_x + \left(1 + \frac{i\nu}{\omega}\right)\mathrm{v}_y = -\frac{ie}{m\omega}E_y \quad \text{(y component)}$$

$$\left(1 + \frac{i\nu}{\omega}\right)\mathrm{v}_z = -\frac{ie}{m\omega}E_z$$

To put this into final form let us replace combinations of numbers by symbols to make manipulation easier. As we will see later all these symbols have a physical meaning.

$$(20) \quad U = \left(1 + \frac{i\nu}{\omega}\right) \quad ; \quad \mathrm{Y} = \frac{\omega_{ce}}{\omega} \quad ; \quad \mathrm{X} = \frac{\omega_{pe}^2}{\omega^2}$$

Y is the ratio of the electron gyration frequency about the magnetic field to the frequency of the whistler wave. X is the ratio (squared of the electron plasma frequency to the wave frequency (note that ω_{pe}^2 is proportional to the plasma density). U tells us how large the collisions are with respect to the whistler wave frequency. Note the i $\left(\sqrt{-1}\right)$ in the equations will cause damping of the wave because collisions take energy from it.

Doing these substitutions:

$$Uv_x + iYv_y = -\frac{ie}{m\omega}E_x$$

$$-iYv_x + Uv_y = -\frac{ie}{m\omega}E_y$$

$$Uv_z = -\frac{ie}{m\omega}E_z$$

and the force equation becomes:

$$(21)\quad \begin{pmatrix} U & iY & 0 \\ -iY & U & 0 \\ 0 & 0 & U \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = -\frac{ie}{m\omega}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

To eliminate v from the right hand side of (19) we have to solve for v above. You can simply divide both sides of the equation by the matrix of U,Y. We have to do a matrix inversion.

Consider a Matrix A

$$\overleftrightarrow{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

The inverted matrix isn't one divided by A it is called A^{-1} and is

$$\overleftrightarrow{A}^{-1} = \frac{1}{\det A}\begin{bmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{bmatrix}$$

A shorthand way of writing this is $\overleftrightarrow{A}^{-1} = \frac{\overleftrightarrow{A}_{cf}^{T}}{\det A}$. Here $\overleftrightarrow{A}_{cf}^{T}$ is the transpose of the co-factors of the matrix. The co-factors are calculated by striking out the row and column of the element we are taking the co-factor of , then taking the determinant of what is left. Also terms from even columns ie A_{21} have a minus sign in front of them. For example for the A_{21} term .

$$cofactor A_{21} = -1 * \det\begin{bmatrix} A_{12} & A_{32} & A_{42} \\ A_{13} & A_{32} & A_{42} \\ A_{14} & A_{34} & A_{44} \end{bmatrix} = -1 * \{A_{12}\left(A_{32}A_{44} - A_{42}A_{34}\right) - A_{32}\left(A_{13}A_{44} - A_{14}A_{42}\right) + A_{42}\left(A_{13}A_{34} - A_{32}A_{14}\right)\}$$

Next you take the transpose of the new matrix which you get by switching indicies: $_{12}$->$_{21}$. We already know how to do the determinant. So next we invert our matrix

(do the algebra and try it!)

$$(22)\quad \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = -\frac{ie}{m\omega U\left(U^2 - Y^2\right)}\begin{pmatrix} U^2 & -iUY & 0 \\ iUY & U^2 & 0 \\ 0 & 0 & \left(U^2 - Y^2\right) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Equation 2 had the v column matrix equal a bunch of stiff and so does equation 19. So we replace v in one to get an equation for E only

$$23)\quad \begin{pmatrix} \left(1-\eta^2\cos^2\theta\right) & 0 & \eta^2\sin\theta\cos\theta \\ 0 & \left(1-\eta^2\right) & 0 \\ \eta^2\sin\theta\cos\theta & 0 & \left(1-\eta^2\sin^2\theta\right) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} =$$

$$\left(\frac{ien}{\varepsilon_0\omega}\right)\left[-\frac{ie}{m\omega U\left(U^2 - Y^2\right)}\right]\begin{pmatrix} U^2 & -iUY & 0 \\ iUY & U^2 & 0 \\ 0 & 0 & \left(U^2 - Y^2\right) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

We then can combine terms by substituting the fact the plasma frequency is $\omega_{pe}^2 = \frac{4\pi ne^2}{m_e}$

$$\begin{pmatrix} \left(1-\eta^2\cos^2\theta\right) & 0 & \eta^2\sin\theta\cos\theta \\ 0 & \left(1-\eta^2\right) & 0 \\ \eta^2\sin\theta\cos\theta & 0 & \left(1-\eta^2\sin^2\theta\right) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{\omega_{pe}^2}{\omega^2 U\left(U^2 - Y^2\right)}\begin{pmatrix} U^2 & -iUY & 0 \\ iUY & U^2 & 0 \\ 0 & 0 & \left(U^2 - Y^2\right) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

and finally

$$\begin{pmatrix} \left(1-\eta^2\cos^2\theta\right) & 0 & \eta^2\sin\theta\cos\theta \\ 0 & \left(1-\eta^2\right) & 0 \\ \eta^2\sin\theta\cos\theta & 0 & \left(1-\eta^2\sin^2\theta\right) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{X}{U\left(U^2-Y^2\right)}\begin{pmatrix} U^2 & -iUY & 0 \\ iUY & U^2 & 0 \\ 0 & 0 & \left(U^2-Y^2\right) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

The next step is to collect all the terms that multiply Ex, Ey, Ez and put them into one big matrix:

(25)

$$\begin{pmatrix} \left(1-\eta^2\cos^2\theta\right)-\dfrac{XU}{\left(U^2-Y^2\right)} & \dfrac{iXY}{\left(U^2-Y^2\right)} & \eta^2\sin\theta\cos\theta \\ -\dfrac{iXY}{\left(U^2-Y^2\right)} & \left(1-\eta^2-\dfrac{XU}{\left(U^2-Y^2\right)}\right) & 0 \\ \eta^2\sin\theta\cos\theta & 0 & \left(1-\eta^2\sin^2\theta-\dfrac{X}{U}\right) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

Are we done yet? Nope. To make the matrix look simpler (of course it isn't any simpler) we define more terms

(26) $S = 1-\dfrac{XU}{U^2-Y^2}$; $D = -\dfrac{XU}{U^2-Y^2}$; $P = 1-\dfrac{X}{U}$

Substituting this into our matrix we arrive at

$$(27)\quad \begin{pmatrix} \left(S-\eta^2\cos^2\theta\right) & -iD & \eta^2\sin\theta\cos\theta \\ iD & \left(S-\eta^2\right) & 0 \\ \eta^2\sin\theta\cos\theta & 0 & \left(P-\eta^2\sin^2\theta\right) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

This is of the form $\overleftrightarrow{M}\vec{E} = 0$. This can only be satisfied if the determinant of M is zero (as it turns out). So next we have to take the determinant of equation (27).
Turn off a video game and try it!

The determinant of a 3X3 matrix has the following form:

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

The answer for our matrix is:

$$\begin{aligned}&\left(S-\eta^2\cos^2\theta\right)\left[\left(S-\eta^2\right)\left(P-\eta^2\sin^2\theta\right)\right]+iD(iD)\left(P-\eta^2\sin^2\theta\right)\\ &\qquad +\eta^2\sin\theta\cos\theta\left(S-\eta^2\right)\eta^2\sin\theta\cos\theta=0\end{aligned}$$

WE go back to our pal algebra and expand this. There are some terms that will go away.

$$\begin{aligned}&\left(S-\eta^2\cos^2\theta\right)\left(SP-S\eta^2\sin^2\theta-P\eta^2+\eta^4\sin^2\theta\right)\\ &-D^2P+D^2\eta^2\sin^2\theta-\eta^4\sin^2\theta\cos^2\theta\left(S-\eta^2\right)=0\end{aligned}$$

Next (note terms that will cancel are in red)

$$\begin{aligned}&\left(S^2P-S^2\eta^2\sin^2\theta-SP\eta^2+S\eta^4\sin^2\theta\right)\\ &-SP\eta^2\cos^2\theta+S\eta^4\sin^2\theta\cos^2\theta+P\eta^4\cos^2\theta-\eta^6\sin^2\theta\cos^2\theta\\ &-D^2P+D^2\eta^2\sin^2\theta-\eta^4S\sin^2\theta\cos^2\theta+\eta^6\sin^2\theta\cos^2\theta=0\end{aligned}$$

Just when you thought we were done lets create more definitions, (Don't lose sight of the fact that we are after η the index of refraction.)

$$R = S + D$$
$$L = S - D$$
or
$$S = \frac{1}{2}(R+L) \quad \text{and} \quad D=\frac{1}{2}(R-L)$$

and $S = 1 - \dfrac{XU}{U^2 - Y^2}$; $D=-\dfrac{XY}{U^2-Y^2}$; $P=1-\dfrac{X}{U}$

We will also need

(28) $$R = S + D = 1 - \frac{XU}{U^2-Y^2} - \frac{XY}{U^2-Y^2} = 1 - \frac{X(U+Y)}{(U+Y)U-Y} = 1 - \frac{X}{U-Y}$$

(29) $L = S - D = 1 - \frac{XU}{U^2 - Y^2} + \frac{XY}{U^2 - Y^2} \;\; =1 - \frac{X(U-Y)}{(U+Y)U-Y} = 1 - \frac{X}{U+Y}$

R and L also have important physical meaning, which we will discuss later.

We can rewrite our combined equation for the index of refraction as

$$\eta^4\left(P\cos^2\theta + S\sin^2\theta\right) -$$
$$\eta^2(S^2\sin^2\theta + SP(1+\cos\theta) - D^2\sin^2\theta) + P\left(S^2 - D^2\right) = 0$$

Then using the expression in (28) and (29) for R and L

$$S^2 - D^2 = \frac{1}{4}(R+L)^2 - \frac{1}{4}(R-L)^2$$
$$S^2 - D^2 = \frac{1}{4}\left[R^2 + 2RL + L^2 - R^2 + 2RL - L^2\right] = RL$$

We finally get the equation for the index of refraction

(30) $\eta^4\left(P\cos^2\theta + S\sin^2\theta\right) - \eta^2(RL\sin^2\theta + SP(1+\cos\theta)) + PRL = 0$

Note it is a 4th degree equation. Is it Appleton's equation. Nope we have more algebra to get there!

We can re-write equation 30 as

(31) $A\eta^4 - B\eta^2 + C = 0$ where you can easily pick A,B,C for equation 30.

Next trick. We can use the quadratic equation solution to solve for η^2

$\eta^2 = \frac{B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$. Put this into equation 31 and we get:

(32) $A\eta^4 - B\eta^2 + A\eta^2 + C = A\eta^2$

This can re-arranged (try it) to get

(33) $\eta^2 = \frac{A\eta^2 - C}{A\eta^2 + A - B} = \frac{\left[B \pm \sqrt{B^2 - 4AC}\right] - 2C}{\left[\pm\sqrt{B^2 - 4AC}\right] + 2A - B}$

and FINALLY

(34) $\eta^2 = 1 - \dfrac{2(A-B+C)}{2A-B\pm\sqrt{B^2-4AC}}$

This is a form of Appleton's equation but not the form we want. To get what we want, guess what, more algebra.

Let us look at the numerator A-B+C. To remind you:

$A=(S\sin^2\theta+P\cos^2\theta$, B=$RL\sin^2\theta+SP\left(1+\cos^2\theta\right)$, C=PRL

$S=1-\dfrac{XU}{U^2-Y^2}$; D=$-\dfrac{XY}{U^2-Y^2}$; P=1-$\dfrac{X}{U}$,$R==1-\dfrac{X}{U-Y}$,$L=1-\dfrac{X}{U+Y}$

Here is the algebra for the numerator. Follow it, note the stuff in red crosses out

For the numerator in (34)

$$A-B+C=\left(S-RL\right)\sin^2\theta+P\cos^2\theta-SP\left(1+\cos^2\theta\right)+PRL$$

$$S-RL=1-\frac{XU}{U^2-Y^2}-\left(1-\frac{X}{U-Y}\right)\left(1-\frac{X}{U+Y}\right)=1-\frac{XU}{U^2-Y^2}-\left(1-\frac{X}{U-Y}-\frac{X}{U+Y}+\frac{X^2}{U^2-Y^2}\right)$$

$$=-\frac{XU}{U^2-Y^2}+\left(\frac{X}{U-Y}+\frac{X}{U+Y}\right)-\frac{X^2}{U^2-Y^2}=-\frac{XU}{U^2-Y^2}+\frac{(XU+XY+XU-XY)}{U^2-Y^2}+\frac{X^2}{U^2-Y^2}$$

$$=\frac{X^2}{U^2-Y^2}+\frac{XU}{U^2-Y^2}=X\frac{\left(U+X\right)}{U^2-Y^2}$$

$$PRL=\left(1-\frac{X}{U}\right)\left(1-\frac{X}{U-Y}\right)\left(1-\frac{X}{U+Y}\right)$$

$$PRL=\left(1-\frac{X}{U}\right)(1-\frac{X}{U-Y}-\frac{X}{U+Y}+\frac{X^2}{U^2-Y^2})$$

$$PRL=1-\frac{X}{U-Y}-\frac{X}{U+Y}+\frac{X^2}{U^2-Y^2}-\frac{X}{U}+\frac{X^2}{U\left(U-Y\right)}+\frac{X^2}{U\left(U+Y\right)}-\frac{X^3}{U(U^2-Y^2)}$$

$$SP=(1-\frac{XU}{U^2-Y^2})(1-\frac{X}{U})=1-\frac{XU}{U^2-Y^2}-\frac{X}{U}+\frac{X^2}{U^2-Y}$$

and

$$A-B+C=X\frac{(U+X)}{U^2-Y^2}\sin^2\theta+\left(1-\frac{X}{U}\right)\cos^2\theta-\left(1-\frac{XU}{U^2-Y^2}-\frac{X}{U}+\frac{X^2}{U^2-Y}\right)\left(1+\cos^2\theta\right)$$

$$+1-\frac{X}{U-Y}-\frac{X}{U+Y}+\frac{X^2}{U^2-Y^2}-\frac{X}{U}+\frac{X^2}{U(U-Y)}+\frac{X^2}{U(U+Y)}-\frac{X^3}{U(U^2-Y^2)}$$

multiply through

$$A-B+C=X\frac{(U-X)}{U^2-Y^2}\sin^2\theta+\left(\cos^2\theta-\frac{X}{U}\cos^2\theta\right)-\left(1-\frac{XU}{U^2-Y^2}-\frac{X}{U}+\frac{X^2}{U^2-Y}\right)$$

$$-\cos^2\theta+\frac{XU}{U^2-Y^2}\cos^2\theta+\frac{X}{U}\cos^2\theta-\frac{X^2}{U^2-Y}\cos^2\theta$$

$$+1-\frac{X}{U-Y}-\frac{X}{U+Y}+\frac{X^2}{U^2-Y^2}-\frac{X}{U}+\frac{X^2}{U(U-Y)}+\frac{X^2}{U(U+Y)}-\frac{X^3}{U(U^2-Y^2)}$$

Finding like terms

$$A-B+C=X\frac{(U-X)}{U^2-Y^2}\sin^2\theta+\left(-\cos^2\theta-\frac{X}{U}\cos^2\theta\right)-1+\frac{XU}{U^2-Y^2}+\frac{X}{U}-\frac{X^2}{U^2-Y}$$

$$+\cos^2\theta+\frac{XU}{U^2-Y^2}\cos^2\theta+\frac{X}{U}\cos^2\theta-\frac{X^2}{U^2-Y}\cos^2\theta+\frac{XU}{U^2-Y^2}$$

$$+1-\frac{2UX}{U^2-Y^2}+\frac{X^2}{U^2-Y^2}-\frac{X}{U}+\frac{X^2}{U(U-Y)}+\frac{X^2}{U(U+Y)}-\frac{X^3}{U(U^2-Y^2)}$$

And then

$$2(A-B+C)=\frac{2X^2}{U^2-Y^2}(1-\frac{X}{U})\ 2(A-B+C)=\frac{2X^2}{U^2-Y^2}(1-\frac{X}{U})$$

This is just the numerator. When you do the denominator one finds it is also proportional to $\frac{1}{U^2-Y^2}$ so that term cancels.

When all the dust settles we finally get

(35)

$$\eta^2=1-\frac{X}{Q}$$

$$Q=U-\frac{Y^2\sin^2\theta}{2(U-X)}\pm\sqrt{\frac{Y^4\sin^4\theta}{4(U-X)^2}+Y^2\cos^2\theta}$$

$$U=\left(1+\frac{i\nu}{\omega}\right)\quad;\quad \mathrm{Y}=\frac{\omega_{ce}}{\omega}\quad;\quad \mathrm{X}=\frac{\omega_{pe}^2}{\omega^2}$$

This is Appleton's equation. Putting in the values for U,Y,X

$$(36)\quad \eta^2 = 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}}{\left(1+\frac{i\nu}{\omega}\right) - \frac{\frac{\omega_{ce}^2}{\omega^2}\sin^2\theta}{2\left(\left(1+\frac{i\nu}{\omega}\right)-\frac{\omega_{pe}^2}{\omega^2}\right)} \pm \sqrt{\frac{\frac{\omega_{ce}^4}{\omega^4}\sin^4\theta}{4\left(\left(1+\frac{i\nu}{\omega}\right)-\frac{\omega_{pe}^2}{\omega^2}\right)^2} +^2 \frac{\omega_{ce}^2}{\omega^2}\cos^2\theta}} \backslash$$

Here the wave damping is the imaginary I times ν. Let us assume there is some damping but it is slight. Then $\left(1+\frac{i\nu}{\omega}\right) - \frac{\omega_{pe}^2}{\omega^2} \cong 1 - \frac{\omega_{pe}^2}{\omega^2}$ This means the frequency associated with the damping is much smaller than the natural plasma frequency. In this situation 36 becomes:

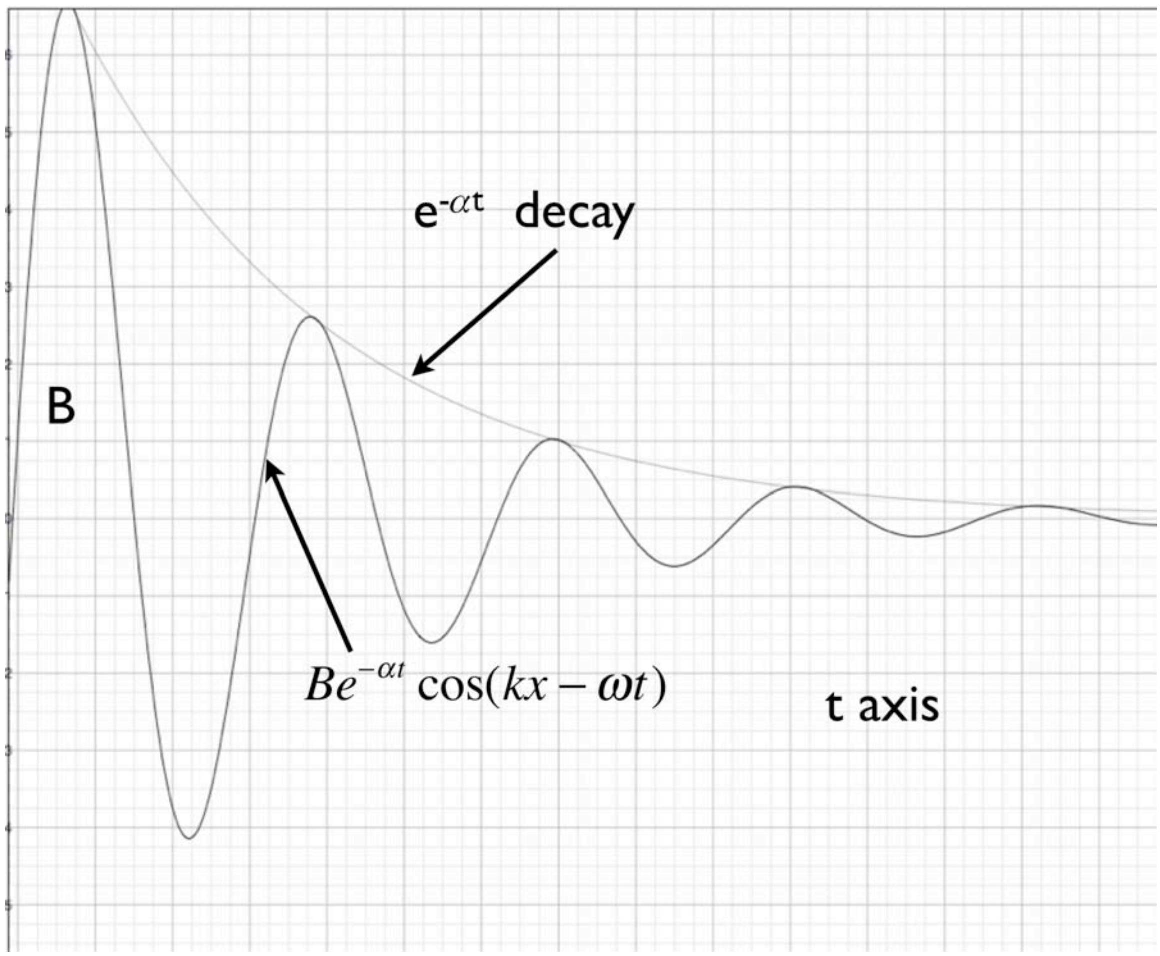
$$(37)\quad \eta^2 = 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}}{\left(1+\frac{i\nu}{\omega}\right) - \frac{\frac{\omega_{ce}^2}{\omega^2}\sin^2\theta}{2\left(1-\frac{\omega_{pe}^2}{\omega^2}\right)} \pm \sqrt{\frac{\left(\frac{\omega_{ce}^2}{\omega^2}\sin^2\theta\right)^2}{4\left(1-\frac{\omega_{pe}^2}{\omega^2}\right)^2} + \frac{\omega_{ce}^2}{\omega^2}\cos^2\theta}}$$

A damping term still remains. What does it mean? The damping gives a total frequency of $\omega' = \omega + i\nu$. If you put this into the general Fourier solution (8)

$$\vec{B}_1(\vec{r},t) = Be^{i(\vec{\mathbf{k}}\bullet\vec{r}-\omega' t)} = Be^{i(\vec{\mathbf{k}}\bullet\vec{r}-\omega t i\nu t)} = Be^{i(\vec{\mathbf{k}}\bullet\vec{r}-\omega t)}e^{-\nu t}$$

When you take the real part to get the cosine solution

$RE(Be^{i(\vec{\mathbf{k}}\bullet\vec{r}-\omega' t)}) = Be^{-\nu t}\cos\left(\vec{\mathbf{k}}\bullet\vec{r}-\omega t\right)$ This is a cosine that decays in time.



Appendix on Fourier Analysis
The following discussion is a short introduction to Fourier analysis. It is assumed the reader knows the fundamentals of complex numbers.

All signals (sound, light, waves, mathematical functions) can be broken down into Fundamental building blocks. These are called the elements of the spectra:

A prism creates a spectrum of colors from white light.

The spectrum has all the component colors or elements that the "white" light is made of.

The prism is a spectrum analyzer for light.

Light is a Wave

The wave is made up of electric and magnetic fields. In vacuum the equation for the propagation of light is a differential equation which looks like:

$$\frac{\partial^2 \vec{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

The light is moving in the z direction at speed c

C = 186,000 mi/sec=3X10^8 meter/s

E is the electric field of the wave

The solution of this equation is very simple:

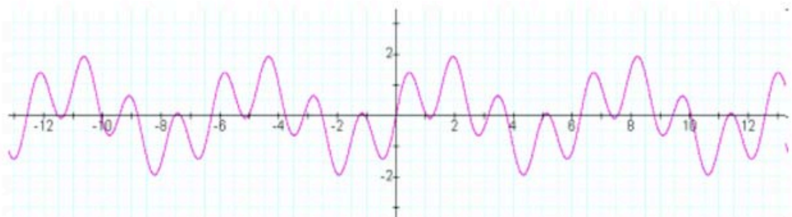
$$E = A\sin\left\{2\pi\left(\frac{z}{\lambda} - \frac{t}{T}\right)\right\} + \cos\left\{2\pi\left(\frac{z}{\lambda} - \frac{t}{T}\right)\right\}$$

With T the wave period, T=1/f and f the frequency

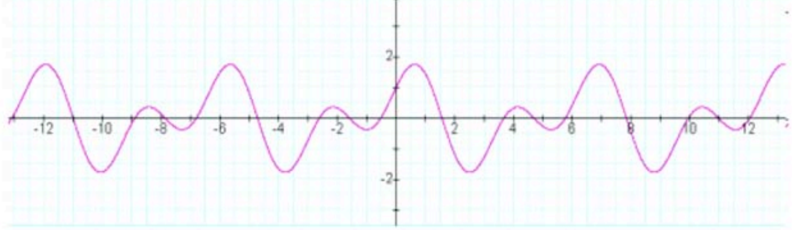
λ is the wavelength (how long from crest to crest) and the wave satisfies

$\lambda f = c$ (This is called the dispersion relation. It relates the frequency and wavelength

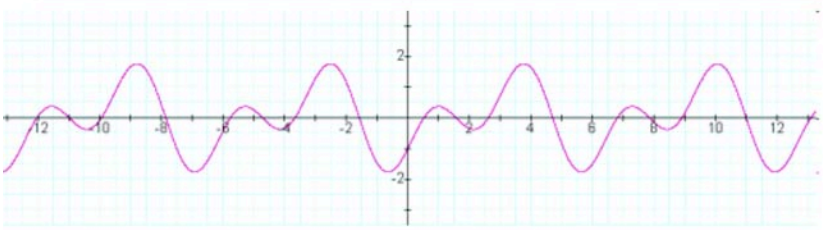
We can add sines and cosines up. Adding them is just like writing z= a+b



y = sinx+sin4x



y = sin2x+cosx



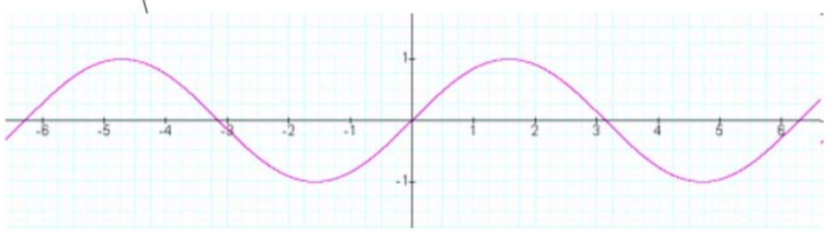
y=sin2x-cosx

To mathematically understand wave and spectra we have to go back to sines and cosines:

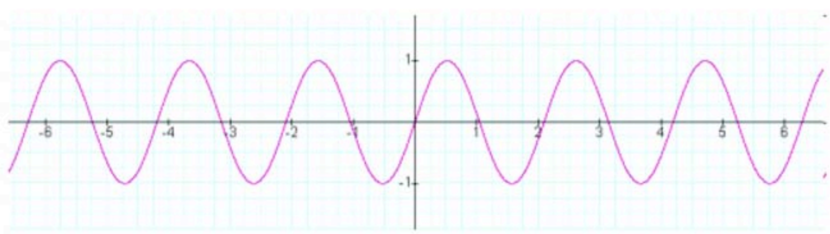
Review: y = sin(at) (the sine)

at is the "argument" of the sine. It is in radians

2π (radians) = 360 (degrees)



a=1

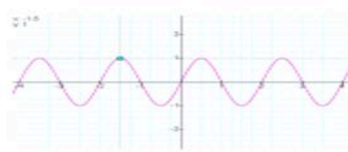


a=3

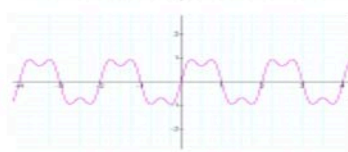
How long is the wavelength here?

$$y = \sum_{k=0}^{n} \frac{\sin((2k+1)\pi x)}{2k+1}$$

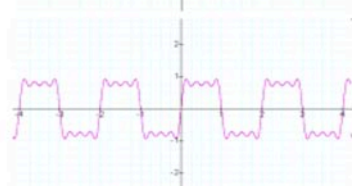
k=0 y1 = sin(πx)



k =1 ; y = y1 + sin(2πx)/3

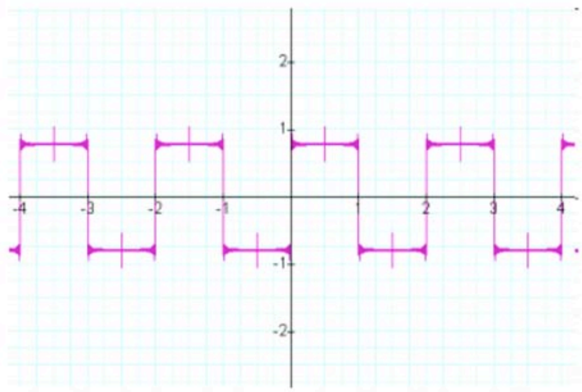


k = 3 ; y = y1 +y2+ sin(7πx)/7



$$y = \sum_{k=0}^{n} \frac{\sin((2k+1)\pi x)}{2k+1}$$

Lets sum up 50 terms!



By summing many sine terms the series converges to a shape that does not look like a sine function . In this case the summation is a square wave. Note that the wave repeats itself with a period of 4.0 units.

Mathematical functions (that are periodic) can all be expressed as an infinite sum of sines and cosines each multiplied by the right constant (like 1/3 or 1/7 as in the previous example)

If F(z) is any periodic function the it can be proven that :

$$F(z) = B_0 + \sum_{m=1}^{\infty} A_m \sin(mkz) + \sum_{m=1}^{\infty} B_m \cos(mkz)$$

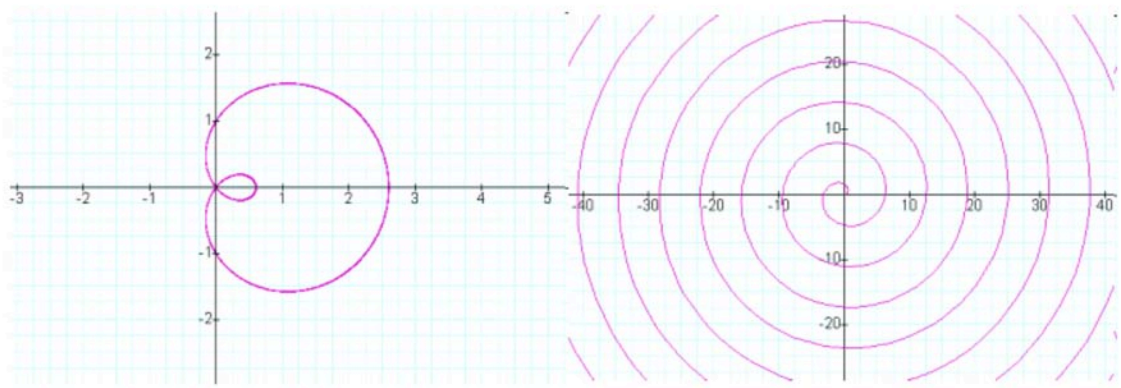
This is called a Fourier Series

This is a function of z which a position like x or y. You can also have a series which describes a function F(t), function of time. The mathematics doesn't care!

Some things can't be expressed as Fourier Series

This spiral is a function you cant use Fourier Series to describe

This is Pascal's snail. No Fourier series here either!



The mathematical form of the Fourier Series is:

$$F(z) = B_0 + \sum_{m=1}^{\infty} A_m \sin(mkz) + \sum_{m=1}^{\infty} B_m \cos(mkz)$$

B_0 is a constant

A_m and B_m are the m_{th} component of the spectrum of the wave. They are called the Fourier coefficients.

Note $k = 2\pi/\lambda$ where λ is the wavelength in the z direction.

What are the A's and B's?

$$B_0 = \frac{1}{\lambda}\int_{z}^{z+\lambda} F(z)dz$$

What do these symbols mean??

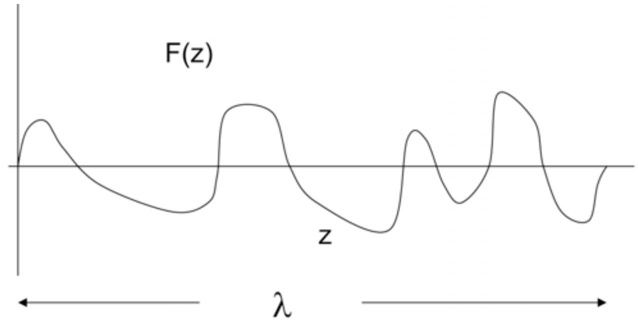
$$A_m = \frac{2}{\lambda}\int_{z}^{z+\lambda} F(z)\sin(mkz)dz$$

$$B_m = \frac{2}{\lambda}\int_{z}^{z+\lambda} F(z)\cos(mkz)dz$$

z is a spatial position, k is a constant. This is a series representing a function that changes in space. Time is fixed.

..

$$A_m = \frac{2}{\lambda}\int_{z}^{z+\lambda} F(z)\sin(mkz)dz$$



This function repeats itself every distance λ along the z axis.

What this "integral" means is for every position z within one wavelength take the function F(z) , which could be an experimental measurement, multiply by the sine:

sin(mkz)= sin(2πmz/λ)

Then go to the next z (which is a very small distance away), multiply by the next sine and add it to the first. This is the same as adding an infinite number of terms together.

The Fourier series for a function which changes in time is :

$$\mathrm{x}(\mathrm{t}) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \quad \omega = 2\pi \mathrm{f} = \frac{2\pi}{\mathrm{T}}$$

$$\mathrm{a_n} = \frac{2}{T}\int_0^T x(t)\cos n\omega t dt \quad \mathrm{n} = 0,1,2,3...$$

$$\mathrm{b_n} = \frac{2}{T}\int_0^T x(t)\sin n\omega t dt \quad \mathrm{n} = 1,2,3...$$

There is an alternate and equivalent way to write x(t)

$$x(t) = a_0 + \sum_{n=1}^{\infty} X_n \cos(n\omega t + \theta_n) \quad \theta_n \text{ is the phase of each term}$$

$$where\ \mathrm{X_n} = \sqrt{a_n^2 + b_n^2} \quad \theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \ \mathrm{n}=1,2,3...$$

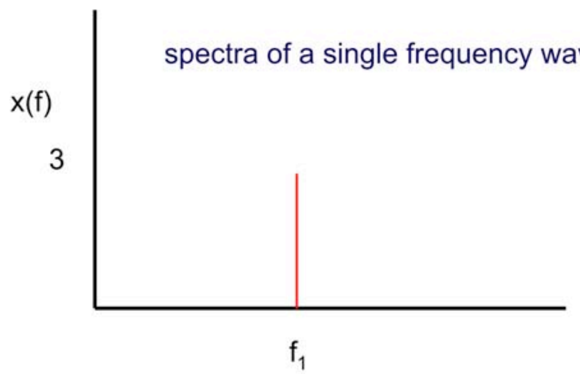
Now lets introduce the concept of spectra

WE know that any time series (such as our data) can be represented by a series:

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \quad \omega = 2\pi f$$

The spectra of x(t) tells us how much of each frequency is present in the signal. Suppose the signal is composed of one sine wave only :

$x(t) = 3\sin(2\pi f_1 t)$



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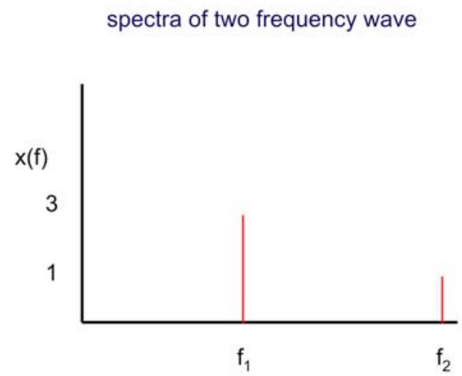
Suppose the signal is composed of two sine waves :

$x(t) = 3\sin(2\pi f_1 t)+\sin(2\pi f_2 t)$; where $f_2 = 3 f_1$



Now there are two lines, one for each frequency

Now suppose that there are many frequencies in our waveform because there are many simultaneous waves in the media