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<u>Cold Plasma Dispersion relation</u>

Let us go back to a single particle and see how it behaves in a high frequency electric field. We will use the force equation and Maxwell's equations. The high frequency field will be that of a wave in the plasma.

The high frequency field is $\vec{E}(t)=\vec{E}_0 e^{i\omega t}$. The frequency can be as high as the cyclotron frequency. The force law is

$\frac{d\vec{\mathrm{v}}}{\mathrm{dt}}=\frac{q}{m}\left(\vec{E}_0 e^{i\omega t}+\vec{\mathrm{v}}\times\vec{B}\right)$. Let $\vec{\mathrm{v}}=\vec{\mathrm{v}}_c+\vec{\mathrm{v}}_E e^{i\omega t}$, where v_c does not depend on ω.

The force law gives us:

$\frac{d\vec{\mathrm{v}}_c}{\mathrm{dt}}+i\omega\vec{\mathrm{v}}_E e^{i\omega t}=\frac{q}{m}\left(\vec{E}_0 e^{i\omega t}+\vec{\mathrm{v}}_c\times\vec{B}+\vec{\mathrm{v}}_E\times\vec{B}e^{i\omega t}\right)$. One set of terms has a ω in front of them all and an $e^{i\omega t}$ dependance, the other does not; in fact we have 2 equations:

$$(I)\quad \frac{d\vec{\mathrm{v}}_c}{\mathrm{dt}}+=\frac{q}{m}\left(\vec{\mathrm{v}}_c\times\vec{B}\right)$$

$$i\omega\vec{\mathrm{v}}_E e^{i\omega t}=\frac{q}{m}\vec{E}_0 e^{i\omega t}+\frac{q}{m}\vec{\mathrm{v}}_E\times\vec{B}e^{i\omega t}$$

The first is the usual cyclotron motion equation, we know the answer(see appendix I) . The second may be re-written as

(II) $(\mathrm{i}\omega+\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{\mathrm{v}}_E=\frac{q}{m}\vec{E}$. Now multiply equation II by the operator $(\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)$

$$\left((\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\right)(\mathrm{i}\omega+\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{\mathrm{v}}_E=\frac{q}{m}(\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{E}.$$

Let us now see what the left hand side is

$$\left((\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\right)(\mathrm{i}\omega+\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{\mathrm{v}}_E=-\omega^2\vec{\mathrm{v}}_E-\frac{q^2}{m^2}\vec{B}\times\left(\vec{B}\times\vec{\mathrm{v}}_E\right)$$

$$=-\omega^2\vec{\mathrm{v}}_E-\frac{q^2}{m^2}\left(\vec{B}\bullet\vec{\mathrm{v}}_E\right)\vec{B}+\frac{q^2}{m^2}B^2\vec{\mathrm{v}}_E$$

Equating both sides

$$-\omega^2\vec{v}_E - \frac{q^2}{m^2}\left(\vec{B}\bullet\vec{v}_E\right)\vec{B} + \frac{q^2}{m^2}B^2\vec{v}_E = \frac{q}{m}(i\omega\text{-}\frac{q}{m}\vec{B}\times)\vec{E}$$ This may be written as

$$\left(\omega_c^2 - \omega^2\right)\vec{v}_E - \frac{q^2}{m^2}\left(\vec{B}\bullet\vec{v}_E\right)\vec{B} = \frac{q}{m}(i\omega\text{-}\frac{q}{m}\vec{B}\times)\vec{E} \;\; ; \;\; \frac{q^2B^2}{m^2} = \omega_c^2$$

The next step is to break the velocity into components perpendicular and parallel to the magnetic field. First for the parallel case. The parallel case $\vec{B}\bullet\vec{v}_{E\|} = Bv_{E_\|}$

$$\vec{v}_E = \vec{v}_{E\|} + \vec{v}_{E\perp}$$

$$\left(\omega_c^2 - \omega^2\right)\vec{v}_{E\|} - \frac{q^2B^2}{m^2}\vec{v}_{E\|} = i\omega\frac{q}{m}\vec{E}_\| \;\; , \;\; \vec{B}\times\vec{E} \text{ is } \perp \text{ to B}$$

(III) $\vec{v}_{E\|} = \text{-}i\frac{q}{\omega m}\vec{E}_\|$ The parallel component of v oscillates as if B was not there but the oscillation is out of phase by 90 degrees ($i = e^{\frac{i\pi}{2}}$). For the perpendicular component

$$\left(\omega_c^2 - \omega^2\right)\vec{v}_{E\perp} = \frac{q}{m}(i\omega\text{-}\vec{\omega}_c\times)\vec{E}_\perp \;\; , \;\; \vec{\omega}_c = \frac{q\vec{B}}{m}$$

(IV) $\vec{v}_{E\perp} = \frac{q}{m}\frac{(i\omega\text{-}\vec{\omega}_c\times)\vec{E}_\perp}{\left(\omega_c^2 - \omega^2\right)}$ Note this has a resonance at the cyclotron frequency. This is an operator equation of the form $\vec{v}_\perp = \vec{A}\vec{E}_\perp$ where A is a complex operator, which could be a tensor.