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<u>Cold Plasma Dispersion relation</u>

Let us go back to a single particle and see how it behaves in a high frequency electric field. We will use the force equation and Maxwell's equations. The high frequency field will be that of a wave in the plasma.

The high frequency field is $\vec{E}(t)=\vec{E}_0 e^{i\omega t}$. The frequency can be as high as the cyclotron frequency. The force law is

$\frac{d\vec{\mathrm{v}}}{\mathrm{dt}}=\frac{q}{m}\left(\vec{E}_0 e^{i\omega t}+\vec{\mathrm{v}}\times\vec{B}\right)$. Let $\vec{\mathrm{v}}=\vec{\mathrm{v}}_{\mathrm{c}}+\vec{\mathrm{v}}_{\mathrm{E}}e^{i\omega t}$, where $\mathrm{v_c}$ does not depend on ω.

The force law gives us:

$\frac{d\vec{\mathrm{v}}_{\mathrm{c}}}{\mathrm{dt}}+i\omega\vec{\mathrm{v}}_{\mathrm{E}}e^{i\omega t}=\frac{q}{m}\left(\vec{E}_0 e^{i\omega t}+\vec{\mathrm{v}}_{\mathrm{c}}\times\vec{B}+\vec{\mathrm{v}}_{\mathrm{E}}\times\vec{B}e^{i\omega t}\right)$. One set of terms has a ω in front of them all and an $e^{i\omega t}$ dependance, the other does not; in fact we have 2 equations:

$$(I)\quad \frac{d\vec{\mathrm{v}}_{\mathrm{c}}}{\mathrm{dt}}+=\frac{q}{m}\left(\vec{\mathrm{v}}_{\mathrm{c}}\times\vec{B}\right)$$

$$i\omega\vec{\mathrm{v}}_{\mathrm{E}}e^{i\omega t}=\frac{q}{m}\vec{E}_0 e^{i\omega t}+\frac{q}{m}\vec{\mathrm{v}}_{\mathrm{E}}\times\vec{B}e^{i\omega t}$$

The first is the usual cyclotron motion equation, we know the answer(see appendix 2) . The second may be re-written as

(2) $(\mathrm{i}\omega+\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{\mathrm{v}}_{\mathrm{E}}=\frac{q}{m}\vec{E}$. Now multiply equation II by the operator

$$(\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)$$

$$\left((\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\right)(\mathrm{i}\omega+\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{\mathrm{v}}_{\mathrm{E}}=\frac{q}{m}(\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{E}.$$

Let us now see what the left hand side is

$$\left((\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\right)(\mathrm{i}\omega+\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{\mathrm{v}}_{\mathrm{E}}=-\omega^2\vec{\mathrm{v}}_{\mathrm{E}}-\frac{q^2}{m^2}\vec{B}\times\left(\vec{B}\times\vec{\mathrm{v}}_{\mathrm{E}}\right)$$

$$=-\omega^2\vec{\mathrm{v}}_{\mathrm{E}}-\frac{q^2}{m^2}\left(\vec{B}\bullet\vec{\mathrm{v}}_{\mathrm{E}}\right)\vec{B}+\frac{q^2}{m^2}B^2\vec{\mathrm{v}}_{\mathrm{E}}$$

Equating both sides

$$-\omega^2\vec{v}_E - \frac{q^2}{m^2}\left(\vec{B}\bullet\vec{v}_E\right)\vec{B} + \frac{q^2}{m^2}B^2\vec{v}_E = \frac{q}{m}(i\omega\text{-}\frac{q}{m}\vec{B}\times)\vec{E}$$ This may be written as

$$\left(\omega_c^2 - \omega^2\right)\vec{v}_E - \frac{q^2}{m^2}\left(\vec{B}\bullet\vec{v}_E\right)\vec{B} = \frac{q}{m}(i\omega\text{-}\frac{q}{m}\vec{B}\times)\vec{E} \; ; \; \frac{q^2B^2}{m^2} = \omega_c^2$$

The next step is to break the velocity into components perpendicular and parallel to the magnetic field. First for the parallel case. The parallel case $\vec{B}\bullet\vec{v}_{E\|} = Bv_{E_\|}$

$$\vec{v}_E = \vec{v}_{E\|} + \vec{v}_{E\perp}$$

$$\left(\omega_c^2 - \omega^2\right)\vec{v}_{E\|} - \frac{q^2B^2}{m^2}\vec{v}_{E\|} = i\omega\frac{q}{m}\vec{E}_\| \quad , \vec{B}\times\vec{E} \text{ is } \perp \text{ to B}$$

(3) $\vec{v}_{E\|} = \text{-}i\frac{q}{\omega m}\vec{E}_\|$ The parallel component of v oscillates as if B was not there but the oscillation is out of phase by 90 degrees ($i = e^{\frac{i\pi}{2}}$).

For the perpendicular component

$$\left(\omega_c^2 - \omega^2\right)\vec{v}_{E\perp} = \frac{q}{m}(i\omega\text{-}\vec{\omega}_c\times)\vec{E}_\perp \quad , \vec{\omega}_c = \frac{q\vec{B}}{m}$$

(4) $\vec{v}_{E\perp} = \frac{q}{m}\frac{(i\omega\text{-}\vec{\omega}_c\times)\vec{E}_\perp}{\left(\omega_c^2 - \omega^2\right)}$ Note this has a resonance at the cyclotron frequency. This is an operator equation of the form $\vec{v}_\perp = \vec{A}\vec{E}_\perp$ where A is a complex operator, which could be a tensor.

Now let us further break down the perpendicular velocity and electric field (which is that of the wave) into two components each rotating around the magnetic field in opposite directions.

$\vec{v}_\perp = \vec{v}_L + \vec{v}_R \quad \vec{E}_\perp = \vec{E}_L + \vec{E}_R$. Using (4) as a guide

(5) $\vec{E}_L \equiv \frac{1}{2}\left[\vec{E}_\perp + \frac{(i\vec{\omega}_c\times)\vec{E}_\perp}{(\omega_c)}\right] ; \vec{E}_R \equiv \frac{1}{2}\left[\vec{E}_\perp - \frac{(i\vec{\omega}_c\times)\vec{E}_\perp}{(\omega_c)}\right] ; \vec{E}_\perp => \vec{E}_\perp e^{i\omega t}$. Let us now assume the magnetic field is constant and is in the z direction.

$$\vec{E}_L = \frac{1}{2}\left[E_\perp e^{i\omega t}\hat{r} + iE_\perp e^{i\omega t}\hat{\theta}\right] ; \vec{E}_R = \frac{1}{2}\left[\left[E_\perp e^{i\omega t}\hat{r} - iE_\perp e^{i\omega t}\hat{\theta}\right]\right] ; \vec{B} = B_0\hat{k}$$

$$\text{Re}(\vec{E}_L) = \frac{1}{2}\left[E_\perp\cos(\omega t)\hat{r} + \text{Re}(i(\cos(\omega t) + i\sin(\omega t)))E_\perp\hat{\theta}\right] = \frac{1}{2}E_\perp\cos(\omega t)\hat{r} - \frac{1}{2}E_\perp\sin(\omega t)E_\perp\hat{\theta}$$

This is an electric field vector that rotates clockwise around the magnetic field. This is the same direction that an ion will take so the E_L field can resonate with the ion gyro motion.
The E_R field will resonant with the electrons as it will rotate in the counterclockwise direction. If we re-write the electric field in the perpendicular direction for ion motion as:

$$\vec{E}_\perp = E_x\hat{i} + E_y\hat{j} \quad \vec{\omega}_C = \omega_C\hat{k}$$

$$for\ \mathrm{E_L}\ ,\ \mathrm{E_L} = \frac{1}{2}\left(E_x\hat{i} + E_y\hat{j}\right) + \frac{1}{2}i\hat{k}\times\left[E_x\hat{i} + E_y\hat{j}\right] = \frac{1}{2}\left(E_x - iE_y\right)\times\left[\hat{i} + i\hat{j}\right]$$

the time dependence is still inside E in the above.
If we put the time dependence back $\mathrm{Re}\left[\hat{i} + i\hat{j}\right]e^{i\omega t} = \cos(\omega t)\hat{i} - \sin(\omega t)\hat{j}$

which is a unit vector spinning in the L direction. Now substitute the rotating vectors into equation (IV) first for E_L then for E_R.

$$(\mathrm{i}\omega\text{-}\vec{\omega}_c\times)\vec{E}_L = \frac{1}{2}\ (\mathrm{i}\omega\text{-}\vec{\omega}_c\times)\left[\vec{E}_\perp + \frac{(\mathrm{i}\vec{\omega}_c\times)\vec{E}_\perp}{(\omega_c)}\right]$$

$$= \frac{1}{2}\{i\omega\vec{E}_\perp - \vec{\omega}_c\times\vec{E}_\perp - \frac{\omega(\vec{\omega}_c\times)\vec{E}_\perp}{(\omega_c)} - \frac{(\mathrm{i}\vec{\omega}_c\times)\left(\vec{\omega}_c\times\vec{E}_\perp\right)}{(\omega_c)}\}$$

$$= \frac{1}{2}i\left\{(\omega+\omega_C)\left[\vec{E}_\perp + \frac{i\vec{\omega}_c\times\vec{E}_\perp}{\omega_c}\right]\right\} = i(\omega+\omega_C)\vec{E}_L \quad (\text{ note } \vec{E}_\perp\bullet\vec{\omega}_c = 0)$$

Then the operator equation (4) is

$$(6)\qquad \begin{aligned} \mathrm{v}_L &= \frac{qi}{m}\frac{(\omega+\omega_c)\vec{E}_L}{\left(\omega_c^2-\omega^2\right)} = \frac{qi}{m}\frac{\vec{E}_L}{(\omega_c-\omega)} \\ \mathrm{v}_R &= \frac{-qi}{m}\frac{\vec{E}_L}{(\omega_c+\omega)} \end{aligned}$$

This may be written as a tensor for the rotating electric field in the frame of the rotating particle

$$(7)\qquad \vec{\mathrm{v}} = \begin{bmatrix}\mathrm{v_L}\\ \mathrm{v_R}\\ \mathrm{v_{\parallel}}\end{bmatrix} = \vec{v}\vec{E} = \frac{iq}{m}\begin{bmatrix}\frac{1}{\omega_c-\omega} & 0 & 0\\ 0 & \frac{-1}{\omega_c+\omega} & 0\\ 0 & 0 & \frac{-1}{\omega}\end{bmatrix}\begin{bmatrix}\mathrm{E_L}\\ \mathrm{E_R}\\ \mathrm{E_{\parallel}}\end{bmatrix}$$

Note that in this rotating frame the mobility tensor is a diagonal. Now using the definitions for E_R, E_L etc in the notes we can transform back into the xyz system (see appendix A) to get:

$$(8)\quad \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \frac{q}{m}\begin{bmatrix} \frac{i\omega}{\omega_c^2-\omega^2} & \frac{\omega_c}{\omega_c^2-\omega^2} & 0 \\ \frac{-\omega_c}{\omega_c^2-\omega^2} & \frac{i\omega}{\omega_c^2-\omega^2} & 0 \\ 0 & 0 & -\frac{i}{\omega} \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

What's the next step? From the particle velocity we can get currents. W can put these into Maxwell's equations and from this we can derive a dispersion relation is we assume wave solutions. Note there is only the velocity of single particles in the wave that we considered so the plasma must be cold.

Consider waves of the form $e^{i(\vec{k}\bullet\vec{r}-\omega t)}$. This means that we have Fourier analyzed the solution t the wave equation. (See the notes on Fourier analysis)

Here is the strategy. We know the relationship between the velocity and electric field. They are related by a tensor

(9) $\vec{v}=\vec{\vec{\nu}}\vec{E}$ The tensor is called the mobility tensor. In the rotating frame $\vec{\vec{\nu}}$ is diagonal

$$(9\text{Rot})\quad \vec{\vec{\nu}}\vec{E} = \frac{iq}{m}\begin{bmatrix} \frac{1}{\omega_c-\omega} & 0 & 0 \\ 0 & \frac{-1}{\omega_c+\omega} & 0 \\ 0 & 0 & -\frac{1}{\omega} \end{bmatrix}$$

In rectangular coordinates $\vec{\vec{\nu}}$ has more components and from (8)

(9Rect) $$\vec{v} = \frac{q}{m}\begin{bmatrix} \frac{i\omega}{\omega_c^2-\omega^2} & \frac{\omega_c}{\omega_c^2-\omega^2} & 0 \\ \frac{-\omega_c}{\omega_c^2-\omega^2} & \frac{i\omega}{\omega_c^2-\omega^2} & 0 \\ 0 & 0 & -\frac{i}{\omega} \end{bmatrix}$$

V and E are also related by Maxwell's equations. One of Maxwell's equation has the current in it. The current is related to the particle drift velocity. (Here we will assume that the current is carried by the electrons to simplify the equations. If you also include ion currents there are additional terms but the physics is the same!) Now we will two relations between the current and electric field and these can be used to find the dielectric tensor.

Of paramount utility in this unravelling is the plasma conductivity σ. It can be shown that if σ -> infinity the magnetic field lines are frozen into the plasma. If σ is not equal to infinity the plasma will dissipate heat like the wires in a toaster. We will consider a plasma with σ finite. The Maxwell equation we will use is

(10) $$\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{j}} + \mu_0 \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

Next we will use Ohms' law. The first time you see Ohm's law in high school it is written as
V=IR. In microscopic form the Voltage , Current and Resistance are represented by

$R \to \rho = \frac{1}{\sigma}$

$V \to \vec{E}$ and $I \to \vec{J}$

Therefore $I = \frac{V}{R} \to \vec{J} = \sigma \vec{E}$

The current and electric field need not be in the same direction. Consider a high frequency field where ω is so large that the ions cannot respond. During an oscillation the electrons can drift in the EXB direction and cause an oscillating current. Thus a field in one direction can cause a current in another, As $\mathrm{v} = \vec{v}\vec{E}$,

$\vec{j} = \sum_{\alpha=i,e} n_\alpha q_\alpha \vec{\mathrm{v}}_\alpha = \vec{\sigma}\vec{E}$. $\vec{\sigma}$ is the conductivity tensor.

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

For example for $j_z = \sigma_{zz} E_z$ this component of the conductivity tensor mediates the current in the z direction cause by an electric field in the z direction, but a current in the z direction can be caused by fields in the other two directions as well. In its most general sense Ohm's law is therefore:

(11) $\vec{J} = \vec{\vec{\sigma}} \vec{E}$

Now using Fourier analysis we time variation of E is $\vec{E} = \vec{E}_0(\vec{r})e^{-i\omega t}$

Equation (10) becomes $\nabla \times \vec{\mathbf{B}} = \mu_0 \left(\vec{\vec{\sigma}} \vec{E} + \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \right)$ and then

(12) $\nabla \times \vec{\mathbf{B}} = \mu_0 \left(\vec{\vec{\sigma}} - i\omega\varepsilon_0 \vec{\vec{1}} \right) \bullet \vec{E}$

We can combine the terms on the right hand side

(13) $\nabla \times \vec{\mathbf{B}} = -i\omega\mu_0 \vec{\vec{\varepsilon}} \bullet \vec{E}$ where $\vec{\vec{\varepsilon}} = \varepsilon_0 \left(\vec{\vec{1}} + \dfrac{i\vec{\vec{\sigma}}}{\omega\varepsilon_0} \right)$ and define a new term

(14) $\left[\dfrac{i\vec{\vec{\sigma}}}{\omega\varepsilon_0} + \vec{\vec{1}} \right] = \vec{\vec{\kappa}}$

But this not so new because $\vec{\vec{\kappa}}$ must be related to the mobility. This is because $\vec{\mathrm{v}} = \vec{\vec{v}} \vec{E}$; $\vec{J} = \vec{\vec{\sigma}} \vec{E}$ and by definition $\vec{J} = nq\vec{\mathrm{v}}$ and Equating these to eliminate E

(15) $\left[\dfrac{inq\vec{\vec{v}}}{\omega\varepsilon_0} + \vec{\vec{1}} \right] = \vec{\vec{\kappa}}$ Here we must be careful and we have to use

$\vec{\vec{1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ when doing the addition since you cab't blithely add a number to a tensor.

also $\omega_{pe} = \sqrt{\frac{nq^2}{m\varepsilon_0}}$ is the definition of the plasma frequency.

$$\vec{\kappa} = \left[\frac{inq\vec{v}}{\omega\varepsilon_0} + \vec{1}\right] = \left[\vec{1} + \frac{inq}{\omega\varepsilon_0}\frac{q}{m}\begin{bmatrix} \frac{i\omega}{\omega_c^2 - \omega^2} & \frac{\omega_c}{\omega_c^2 - \omega^2} & 0 \\ \frac{-\omega_c}{\omega_c^2 - \omega^2} & \frac{i\omega}{\omega_c^2 - \omega^2} & 0 \\ 0 & 0 & -\frac{i}{\omega} \end{bmatrix}\right]$$

(16) $$\vec{\kappa} = \begin{bmatrix} 1 - \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} & \frac{-i\omega_c\omega_{pe}^2}{\omega\left(\omega_{ce}^2 - \omega^2\right)^2} & 0 \\ \frac{i\omega_c\omega_{pe}^2}{\omega\left(\omega_{ce}^2 - \omega^2\right)} & 1 - \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_{pe}^2}{\omega^2} \end{bmatrix}$$

<u>Appendix A</u>

How do we go from rotating coordinates back to rectangular ones? In the notes:

$$\text{(A1)}\quad \vec{E}_L = \frac{1}{2}\left(E_x - iE_y\right)\left(\hat{i} + i\hat{j}\right)$$

What about the other component?

$$\text{(A2)}\quad \vec{E}_R = \frac{1}{2}\left(E_\perp - \frac{i\vec{\omega}_c \times \vec{E}_\perp}{\omega_c}\right),$$ substitute for $E_\perp$ in this

$$\vec{E}_R = \frac{1}{2}\left(E_x\hat{i} + E_y\hat{j} - \frac{i\vec{\omega}_c \times \left(E_x\hat{i} + E_y\hat{j}\right)}{\omega_c}\right) = \frac{1}{2}\left(E_x\hat{i} + E_y\hat{j} - \frac{i\omega_c\hat{k} \times \left(E_x\hat{i} + E_y\hat{j}\right)}{\omega_c}\right)$$

$$\vec{E}_r = \frac{1}{2}\left(E_x\hat{i} + E_y\hat{j} - \frac{i\omega_c\left(E_x\hat{j} - E_y\hat{i}\right)}{\omega_c}\right) = \frac{1}{2}\left(E_x\left[\hat{i} - i\hat{j}\right] + E_y\left[\hat{j} + i\hat{i}\right]\right)$$

$$\text{(A3)}\quad \vec{E}_r = \frac{1}{2}\left(E_x + iE_y\right)\left[\hat{i} - i\hat{j}\right]$$

Now with (A1) and (A3) we have to find E_x, E_y in terms of the left and right components. The velocities have the same form as the fields

$$\text{(A4)}\quad \begin{aligned} \vec{v}_L &= \frac{1}{2}\left(v_x - iv_y\right)\left[\hat{i} + i\hat{j}\right] \\ \vec{v}_R &= \frac{1}{2}\left(v_x + iv_y\right)\left[\hat{i} - i\hat{j}\right] \end{aligned}$$

But $\vec{v}_L = \frac{iq}{m}\frac{1}{\left(\omega_c - \omega\right)}\vec{E}_L$

Therefore by substitution:

$$\left(v_x - iv_y\right) = \frac{iq}{m}\frac{1}{\left(\omega_c - \omega\right)}\frac{1}{2}\left(E_x - iE_y\right)$$

$$\text{(A5)}\quad v_y = -\frac{q}{m}\frac{1}{\left(\omega_c - \omega\right)}\left(E_x - iE_y\right) - iv_x$$

Next use the definition of v_r

$$\vec{v}_R = -\frac{iq}{m}\frac{1}{\left(\omega_c + \omega\right)}\vec{E}_R$$

and using the definitions again

$$\frac{1}{2}\left(\mathrm{v}_x+i\mathrm{v}_y\right)=-\frac{iq}{m}\frac{1}{\left(\omega_c+\omega\right)}\frac{1}{2}\left(E_x+iE_y\right)$$

$$i\mathrm{v}_y=-\frac{iq}{m}\frac{1}{\left(\omega_c+\omega\right)}\left(E_x+iE_y\right)-\mathrm{v}_x$$

$$\mathrm{v}_y=-\frac{q}{m}\frac{1}{\left(\omega_c+\omega\right)}\left(E_x+iE_y\right)+i\mathrm{v}_x \qquad \text{(A6)}$$

Next equate the two (A5) and (A6)

$$-\frac{q}{m}\frac{1}{\left(\omega_c+\omega\right)}\left(E_x+iE_y\right)+i\mathrm{v}_x=-\frac{q}{m}\frac{1}{\left(\omega_c-\omega\right)}\left(E_x-iE_y\right)-i\mathrm{v}_x$$

$$2i\mathrm{v}_x=-\frac{q}{m}\frac{1}{\left(\omega_c-\omega\right)}\left(E_x-iE_y\right)+\frac{q}{m}\frac{1}{\left(\omega_c+\omega\right)}\left(E_x+iE_y\right)$$

$$2i\mathrm{v}_x=\frac{q}{m}\left[-\frac{1}{\left(\omega_c-\omega\right)}+\frac{1}{\left(\omega_c+\omega\right)}\right]E_x+iE_y\frac{q}{m}\left[\frac{1}{\left(\omega_c+\omega\right)}+\frac{1}{\left(\omega_c-\omega\right)}\right]$$

$$\mathrm{v}_x=\frac{q}{m}\left[\frac{i\omega}{\left(\omega_c^2-\omega^2\right)}\right]E_x+\frac{q}{m}\left[\frac{\omega_c}{\left(\omega_c^2-\omega^2\right)}\right]E_y$$

Do the same thing for the y component to get

$$\mathrm{v}_y=\frac{q}{m}\left[\frac{-\omega_c}{\left(\omega_c^2-\omega^2\right)}\right]E_x+\frac{q}{m}\left[\frac{i\omega}{\left(\omega_c^2-\omega^2\right)}\right]E_y$$

From before the z component was simple

$$\mathrm{v}_{\parallel}=\mathrm{v}_z=-\frac{iq}{m\omega}E_z$$

We can put this into matrix form

$$\begin{pmatrix}\mathrm{v}_x\\ \mathrm{v}_y\\ \mathrm{v}_z\end{pmatrix}=\frac{q}{m}\begin{bmatrix}\frac{i\omega}{\left(\omega_c^2-\omega^2\right)} & \frac{\omega_c}{\left(\omega_c^2-\omega^2\right)} & 0\\ \frac{-\omega_c}{\left(\omega_c^2-\omega^2\right)} & \frac{i\omega}{\left(\omega_c^2-\omega^2\right)} & 0\\ 0 & 0 & \frac{i}{\omega}\end{bmatrix}\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix}$$