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Summer 2011

Cold Plasma Dispersion relation

Let us go back to a single particle and see how it behaves in a high frequency electric field. We will use the force equation and Maxwell's equations. The high frequency field will be that of a wave in the plasma.

The high frequency field is $\vec{E}(t)=\vec{E}_0 e^{i\omega t}$. The frequency can be as high as the cyclotron frequency. The force law is

$\frac{d\vec{\mathrm{v}}}{\mathrm{dt}}=\frac{q}{m}\left(\vec{E}_0 e^{i\omega t}+\vec{\mathrm{v}}\times\vec{B}\right)$. Let $\vec{\mathrm{v}}=\vec{\mathrm{v}}_{\mathrm{c}}+\vec{\mathrm{v}}_{\mathrm{E}} e^{i\omega t}$, where v_{c} does not depend on ω.

The force law gives us:

$\frac{d\vec{\mathrm{v}}_{\mathrm{c}}}{\mathrm{dt}}+i\omega\vec{\mathrm{v}}_{\mathrm{E}} e^{i\omega t}=\frac{q}{m}\left(\vec{E}_0 e^{i\omega t}+\vec{\mathrm{v}}_{\mathrm{c}}\times\vec{B}+\vec{\mathrm{v}}_{\mathrm{E}}\times\vec{B}e^{i\omega t}\right)$. One set of terms has a ω in front of them all and an $e^{i\omega t}$ dependance, the other does not; in fact we have 2 equations:

$$(I)\quad \frac{d\vec{\mathrm{v}}_{\mathrm{c}}}{\mathrm{dt}}+=\frac{q}{m}\left(\vec{\mathrm{v}}_{\mathrm{c}}\times\vec{B}\right)$$

$$i\omega\vec{\mathrm{v}}_{\mathrm{E}} e^{i\omega t}=\frac{q}{m}\vec{E}_0 e^{i\omega t}+\frac{q}{m}\vec{\mathrm{v}}_{\mathrm{E}}\times\vec{B}e^{i\omega t}$$

The first is the usual cyclotron motion equation, we know the answer(see appendix I) . The second may be re-written as

(II) $(\mathrm{i}\omega+\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{\mathrm{v}}_{\mathrm{E}}=\frac{q}{m}\vec{E}$. Now multiply equation II by the operator

$$(\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)$$

$$\left((\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\right)(\mathrm{i}\omega+\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{\mathrm{v}}_{\mathrm{E}}=\frac{q}{m}(\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{E}.$$

Let us now see what the left hand side is

$$\left((\mathrm{i}\omega-\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\right)(\mathrm{i}\omega+\frac{\mathrm{q}}{\mathrm{m}}\vec{B}\times)\vec{\mathrm{v}}_{\mathrm{E}}=-\omega^2\vec{\mathrm{v}}_{\mathrm{E}}-\frac{q^2}{m^2}\vec{B}\times\left(\vec{B}\times\vec{\mathrm{v}}_{\mathrm{E}}\right)$$

$$=-\omega^2\vec{\mathrm{v}}_{\mathrm{E}}-\frac{q^2}{m^2}\left(\vec{B}\bullet\vec{\mathrm{v}}_{\mathrm{E}}\right)\vec{B}+\frac{q^2}{m^2}B^2\vec{\mathrm{v}}_{\mathrm{E}}$$

Equating both sides

$-\omega^2 \vec{v}_E - \frac{q^2}{m^2}\left(\vec{B}\bullet\vec{v}_E\right)\vec{B} + \frac{q^2}{m^2}B^2\vec{v}_E = \frac{q}{m}(i\omega - \frac{q}{m}\vec{B}\times)\vec{E}$ This may be written as

$$\left(\omega_c^2 - \omega^2\right)\vec{v}_E - \frac{q^2}{m^2}\left(\vec{B}\bullet\vec{v}_E\right)\vec{B} = \frac{q}{m}(i\omega - \frac{q}{m}\vec{B}\times)\vec{E} \;\; ; \;\; \frac{q^2B^2}{m^2} = \omega_c^2$$

The next step is to break the velocity into components perpendicular and parallel to the magnetic field. First for the parallel case. The parallel case $\vec{B}\bullet\vec{v}_{E\|} = Bv_{E_\|}$

$$\vec{v}_E = \vec{v}_{E\|} + \vec{v}_{E\perp}$$

$$\left(\omega_c^2 - \omega^2\right)\vec{v}_{E\|} - \frac{q^2B^2}{m^2}\vec{v}_{E\|} = i\omega\frac{q}{m}\vec{E}_\| \;\; , \;\; \vec{B}\times\vec{E} \text{ is } \perp \text{ to B}$$

(III) $\vec{v}_{E\|} = -i\frac{q}{\omega m}\vec{E}_\|$ The parallel component of v oscillates as if B was not there but the oscillation is out of phase by 90 degrees ($i = e^{\frac{i\pi}{2}}$).

For the perpendicular component

$$\left(\omega_c^2 - \omega^2\right)\vec{v}_{E\perp} = \frac{q}{m}(i\omega - \vec{\omega}_c\times)\vec{E}_\perp \;\; , \;\; \vec{\omega}_c = \frac{q\vec{B}}{m}$$

(IV) $\vec{v}_{E\perp} = \frac{q}{m}\frac{(i\omega - \vec{\omega}_c\times)\vec{E}_\perp}{\left(\omega_c^2 - \omega^2\right)}$ Note this has a resonance at the cyclotron frequency. This is an operator equation of the form $\vec{v}_\perp = \vec{A}\vec{E}_\perp$ where A is a complex operator, which could be a tensor.

Now let us further break down the perpendicular velocity and electric field (which is that of the wave) into two components each rotating around the magnetic field in opposite directions.

$\vec{v}_\perp = \vec{v}_L + \vec{v}_R \quad \vec{E}_\perp = \vec{E}_L + \vec{E}_R$. Using (IV) as a guide

(V) $\vec{E}_L \equiv \frac{1}{2}\left[\vec{E}_\perp + \frac{(i\vec{\omega}_c\times)\vec{E}_\perp}{(\omega_c)}\right] ; \vec{E}_R \equiv \frac{1}{2}\left[\vec{E}_\perp - \frac{(i\vec{\omega}_c\times)\vec{E}_\perp}{(\omega_c)}\right] ; \vec{E}_\perp \Rightarrow \vec{E}_\perp e^{i\omega t}$. Let us now assume the magnetic field is constant and is in the z direction.

$$\vec{E}_L = \frac{1}{2}\left[E_\perp e^{i\omega t}\hat{r} + iE_\perp e^{i\omega t}\hat{\theta}\right] ; \vec{E}_R = \frac{1}{2}\left[\left[E_\perp e^{i\omega t}\hat{r} - iE_\perp e^{i\omega t}\hat{\theta}\right]\right] ; \vec{B} = B_0\hat{k}$$

$$\mathrm{Re}(\vec{E}_L) = \frac{1}{2}\left[E_\perp\cos(\omega t)\hat{r} + \mathrm{Re}(i(\cos(\omega t) + i\sin(\omega t)))E_\perp\hat{\theta}\right] = \frac{1}{2}E_\perp\cos(\omega t)\hat{r} - \frac{1}{2}E_\perp\sin(\omega t)E_\perp\hat{\theta}$$

This is an electric field vector that rotates clockwise around the magnetic field. This is the same direction that an ion will take so the E_L field can resonate with the ion gyro motion.
The E_R field will resonant with the electrons as it will rotate in the counterclockwise direction. If we re-write the electric field in the perpendicular direction for ion motion as:

$$\vec{E}_\perp = E_x\hat{i} + E_y\hat{j} \quad \vec{\omega}_C = \omega_C\hat{k}$$

$$for\ \mathrm{E_L}\ ,\ \mathrm{E_L} = \frac{1}{2}\left(E_x\hat{i} + E_y\hat{j}\right) + \frac{1}{2}i\hat{k}\times\left[E_x\hat{i} + E_y\hat{j}\right] = \frac{1}{2}\left(E_x - iE_y\right)\times\left[\hat{i} + i\hat{j}\right]$$

the time dependence is still inside E in the above.
If we put the time dependence back $\mathrm{Re}\left[\hat{i} + i\hat{j}\right]e^{i\omega t} = \cos(\omega t)\hat{i} - \sin(\omega t)\hat{j}$

which is a unit vector spinning in the L direction. Now substitute the rotating vectors into equation (IV) first for E_L then for E_R.

$$(\mathrm{i}\omega\text{-}\vec{\omega}_c\times)\vec{E}_L = \frac{1}{2}\ (\mathrm{i}\omega\text{-}\vec{\omega}_c\times)\left[\vec{E}_\perp + \frac{(\mathrm{i}\vec{\omega}_c\times)\vec{E}_\perp}{(\omega_c)}\right]$$

$$= \frac{1}{2}\{i\omega\vec{E}_\perp - \vec{\omega}_c\times\vec{E}_\perp - \frac{\omega(\vec{\omega}_c\times)\vec{E}_\perp}{(\omega_c)} - \frac{(\mathrm{i}\vec{\omega}_c\times)\left(\vec{\omega}_c\times\vec{E}_\perp\right)}{(\omega_c)}\}$$

$$= \frac{1}{2}i\left\{(\omega+\omega_C)\left[\vec{E}_\perp + \frac{i\vec{\omega}_c\times\vec{E}_\perp}{\omega_c}\right]\right\} = i(\omega+\omega_C)\vec{E}_L \quad (\text{ note } \vec{E}_\perp\bullet\vec{\omega}_c = 0)$$

Then the operator equation (IV) is

$$\text{(V)}\qquad \mathrm{v}_L = \frac{qi}{m}\frac{(\omega+\omega_c)\vec{E}_L}{\left(\omega_c^2-\omega^2\right)} = \frac{qi}{m}\frac{\vec{E}_L}{(\omega_c-\omega)} \qquad \mathrm{v}_R = \frac{-qi}{m}\frac{\vec{E}_L}{(\omega_c+\omega)}$$

This may be written as a tensor for the rotating electric field in the frame of the rotating particle

$$\text{(VI)}\qquad \vec{\mathrm{v}} = \begin{bmatrix}\mathrm{v_L}\\ \mathrm{v_R}\\ \mathrm{v_{\parallel}}\end{bmatrix} = \vec{v}\vec{E} = \frac{iq}{m}\begin{bmatrix}\frac{1}{\omega_c-\omega} & 0 & 0\\ 0 & \frac{-1}{\omega_c+\omega} & 0\\ 0 & 0 & \frac{-1}{\omega}\end{bmatrix}\begin{bmatrix}\mathrm{E_L}\\ \mathrm{E_R}\\ \mathrm{E_{\parallel}}\end{bmatrix}$$

Note that in this rotating frame the mobility tensor is a diagonal. Now using the definitions for E_R, E_L etc in the notes we can transform back into the xyz system (see appendix A) to get:

$$\text{(VII)} \quad \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \frac{q}{m} \begin{bmatrix} \dfrac{i\omega}{\omega_c^2 - \omega^2} & \dfrac{\omega_c}{\omega_c^2 - \omega^2} & 0 \\ \dfrac{-\omega_c}{\omega_c^2 - \omega^2} & \dfrac{i\omega}{\omega_c^2 - \omega^2} & 0 \\ 0 & 0 & -\dfrac{i}{\omega} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Appendix A

How do we go from rotating coordinates back to rectangular ones? In the notes:

(A1) $\vec{E}_L = \frac{1}{2}\left(E_x - iE_y\right)\left(\hat{i} + i\hat{j}\right)$

What about the other component?

(A2) $\vec{E}_R = \frac{1}{2}\left(E_\perp - \frac{i\vec{\omega}_c \times \vec{E}_\perp}{\omega_c}\right)$, substitute for $E_\perp$ in this

$$\vec{E}_R = \frac{1}{2}\left(E_x\hat{i} + E_y\hat{j} - \frac{i\vec{\omega}_c \times \left(E_x\hat{i} + E_y\hat{j}\right)}{\omega_c}\right) = \frac{1}{2}\left(E_x\hat{i} + E_y\hat{j} - \frac{i\omega_c\hat{k} \times \left(E_x\hat{i} + E_y\hat{j}\right)}{\omega_c}\right)$$

$$\vec{E}_r = \frac{1}{2}\left(E_x\hat{i} + E_y\hat{j} - \frac{i\omega_c\left(E_x\hat{j} - E_y\hat{i}\right)}{\omega_c}\right) = \frac{1}{2}\left(E_x\left[\hat{i} - i\hat{j}\right] + E_y\left[\hat{j} + i\hat{i}\right]\right)$$

(A3) $\vec{E}_r = \frac{1}{2}\left(E_x + iE_y\right)\left[\hat{i} - i\hat{j}\right]$

Now with (A1) and (A3) we have to find E_x, E_y in terms of the left and right components. The velocities have the same form as the fields

(A4)
$$\vec{\mathrm{v}}_L = \frac{1}{2}\left(\mathrm{v}_x - i\mathrm{v}_y\right)\left[\hat{i} + i\hat{j}\right]$$
$$\vec{\mathrm{v}}_R = \frac{1}{2}\left(\mathrm{v}_x + i\mathrm{v}_y\right)\left[\hat{i} - i\hat{j}\right]$$

But $\vec{\mathrm{v}}_L = \frac{iq}{m}\frac{1}{(\omega_c - \omega)}\vec{E}_L$

Therefore by substitution:

$$\left(\mathrm{v}_x - i\mathrm{v}_y\right) = \frac{iq}{m}\frac{1}{(\omega_c - \omega)}\frac{1}{2}\left(E_x - iE_y\right)$$

$$\text{(A5)} \quad \mathrm{v}_y = -\frac{q}{m}\frac{1}{(\omega_c - \omega)}\left(E_x - iE_y\right) - i\mathrm{v}_x$$

Next use the definition of v_r

$$\vec{\mathrm{v}}_R = -\frac{iq}{m}\frac{1}{(\omega_c + \omega)}\vec{E}_R$$

and using the definitions again

$$\frac{1}{2}\left(\mathrm{v}_x + i\mathrm{v}_y\right) = -\frac{iq}{m}\frac{1}{(\omega_c + \omega)}\frac{1}{2}\left(E_x + iE_y\right)$$

$$i\mathrm{v}_y = -\frac{iq}{m}\frac{1}{(\omega_c + \omega)}\left(E_x + iE_y\right) - \mathrm{v}_x$$

$$\mathrm{v}_y = -\frac{q}{m}\frac{1}{(\omega_c + \omega)}\left(E_x + iE_y\right) + i\mathrm{v}_x \qquad \text{(A6)}$$

Next equate the two (A5) and (A6)

$$-\frac{q}{m}\frac{1}{(\omega_c + \omega)}\left(E_x + iE_y\right) + i\mathrm{v}_x = -\frac{q}{m}\frac{1}{(\omega_c - \omega)}\left(E_x - iE_y\right) - i\mathrm{v}_x$$

$$2i\mathrm{v}_x = -\frac{q}{m}\frac{1}{(\omega_c - \omega)}\left(E_x - iE_y\right) + \frac{q}{m}\frac{1}{(\omega_c + \omega)}\left(E_x + iE_y\right)$$

$$2i\mathrm{v}_x = \frac{q}{m}\left[-\frac{1}{(\omega_c - \omega)} + \frac{1}{(\omega_c + \omega)}\right]E_x + iE_y\frac{q}{m}\left[\frac{1}{(\omega_c + \omega)} + \frac{1}{(\omega_c - \omega)}\right]$$

$$\mathrm{v}_x = \frac{q}{m}\left[\frac{i\omega}{(\omega_c^2 - \omega^2)}\right]E_x + \frac{q}{m}\left[\frac{\omega_c}{(\omega_c^2 - \omega^2)}\right]E_y$$

Do the same thing for the y component to get

$$\mathrm{v}_y = \frac{q}{m}\left[\frac{-\omega_c}{(\omega_c^2 - \omega^2)}\right]E_x + \frac{q}{m}\left[\frac{i\omega}{(\omega_c^2 - \omega^2)}\right]E_y$$

From before the z component was simple

$$v_{\parallel} = v_z = -\frac{iq}{m\omega} E_z$$

We can put this into matrix form

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \frac{q}{m} \begin{bmatrix} \frac{i\omega}{\left(\omega_c^2 - \omega^2\right)} & \frac{\omega_c}{\left(\omega_c^2 - \omega^2\right)} & 0 \\ \frac{-\omega_c}{\left(\omega_c^2 - \omega^2\right)} & \frac{i\omega}{\left(\omega_c^2 - \omega^2\right)} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$