Calculation of Magnetic Field Inside Plasma Chamber

Kevin Kuns

August 14, 2007

Abstract

The calculation of the magnetic field due to a current loop is developed by first finding the magnetic vector potential and then taking its divergence. The solution for all points in space requires the use of elliptic integrals; however, it is shown that for points on the axis of the loop, this solution is identical to the simpler algebraic solution for points located on the axis. The specific magnetic field of the UCLA high school plasma machine is calculated by adding the effects of the 120 current loops that generate the magnetic field of the machine.

1 Elliptic Integrals

In this section we explore two elliptic integrals and their derivatives that will be important in the calculation of the magnetic field.

The complete elliptic integral of the first kind $K(k)$ is defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} \tag{1}$$

and the complete elliptic integral of the second kind $E(k)$ is defined as

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \alpha}\, d\alpha. \tag{2}$$

It is clear from these definitions that

$$K(0) = E(0) = \int_0^{\frac{\pi}{2}} d\alpha = \frac{\pi}{2}. \tag{3}$$

The derivative of K is given by

$$\frac{dK}{dk} = \int_0^{\frac{\pi}{2}} \frac{k\sin^2\alpha}{(1-k^2\sin^2\alpha)^{\frac{3}{2}}} d\alpha. \tag{4}$$

Since

$$\frac{k\sin^2\alpha}{(1-k^2\sin^2\alpha)^{\frac{3}{2}}} = \frac{1}{k(1-k^2\sin^2\alpha)^{\frac{3}{2}}} - \frac{1}{k\sqrt{1-k^2\sin^2\alpha}},$$

(4) can be written as

$$\frac{dK}{dk} = \int_0^{\frac{\pi}{2}} \frac{1}{k(1-k^2\sin^2\alpha)^{\frac{3}{2}}} d\alpha - \int_0^{\frac{\pi}{2}} \frac{1}{k\sqrt{1-k^2\sin^2\alpha}} d\alpha. \tag{5}$$

By (1) the second integral is $\frac{K}{k}$. It can be shown by comparing power series, see [1], that

$$E(k) = (1-k^2)\int_0^{\frac{\pi}{2}} \frac{d\alpha}{(1-k^2\sin^2\alpha)^{\frac{3}{2}}}$$

and therefore (5) becomes

$$\frac{dK}{dk} = \frac{E}{k(1-k^2)} - \frac{K}{k}. \tag{6}$$

The derivative of E is given by

$$\frac{dE}{dk} = -\int_0^{\frac{\pi}{2}} \frac{k\sin^2\alpha}{\sqrt{1-k^2\sin^2\alpha}}\, d\alpha. \tag{7}$$

Since

$$-\frac{k\sin^2\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \frac{\sqrt{1-k^2\sin^2\alpha}}{k} - \frac{1}{k\sqrt{1-k^2\sin^2\alpha}},$$

(7) can be written as

$$\frac{dE}{dk} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{1-k^2\sin^2\alpha}}{k}\, d\alpha - \int_0^{\frac{\pi}{2}} \frac{d\alpha}{k\sqrt{1-k^2\sin^2\alpha}}$$

which by (1) and (2) becomes

$$\frac{dE}{dk} = \frac{E}{k} - \frac{K}{k}. \tag{8}$$

2 Magnetic Field of a Current Loop

In this section we derive the equation for the magnetic field due to a circular loop of current. We will first find the vector potential $\mathbf{A}$ from

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{s}}{R}. \tag{9}$$

We will then obtain the magnetic field $\mathbf{B}$ by taking the curl of $\mathbf{A}$ according to

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{10}$$

2.1 Magnetic Vector Potential

We begin by finding the magnetic vector potential of a loop of wire with radius a by the methods of [2]. We let the center of the loop be on the z axis with the loop in a plane parallel to the xy plane and with a distance h from the x axis. We will find $\mathbf{A}$ from (9) at a point P in the xz plane with coordinates $(r, 0, z)$.

If we let ϕ be the angle from the x axis to any point on the loop the coordinates of that point are $(a\cos\phi, a\sin\phi, h)$. The distance from P to any point on the loop is therefore given by

$$\begin{aligned} R &= \sqrt{(r - a\cos\phi)^2 + a^2\sin^2\phi + (z-h)^2} \\ &= \sqrt{r^2 - 2ar\cos\phi + a^2(\cos^2\phi + \sin^2\phi) + (z-h)^2} \\ &= \sqrt{r^2 + a^2 + (z-h)^2 - 2ar\cos\phi}. \end{aligned}$$

Since

$$d\mathbf{s} = -a\sin\phi\; d\phi\hat{\mathbf{x}} + a\cos\phi\; d\phi\hat{\mathbf{y}},$$

from (9) we have

$$\begin{aligned} \mathbf{A} = \frac{\mu_0 I}{4\pi}\Bigg[\int_0^{2\pi} & \frac{-a\sin\phi\; d\phi}{\sqrt{r^2 + a^2 + (z-h)^2 - 2ar\cos\phi}}\hat{\mathbf{x}} + \\ & \int_0^{2\pi} \frac{a\cos\phi\; d\phi}{\sqrt{r^2 + a^2 + (z-h)^2 - 2ar\cos\phi}}\hat{\mathbf{y}}\Bigg]. \end{aligned} \tag{11}$$

For every ϕ the contribution made to $\mathbf{A}$ in the x direction is canceled by the contribution from $-\phi$. Therefore, there is no x component to $\mathbf{A}$ and the first term in (11) goes to zero. Furthermore the contribution from ϕ in the y direction from ϕ is equal to the contribution from $-\phi$. Therefore, instead of integrating the y component of $\mathbf{A}$ from 0 to 2π we can integrate from 0 to π and multiply by 2. Finally, if we switch to cylindrical coordinates the y component becomes the ϕ component and (11) becomes

$$\mathbf{A} = \frac{\mu_0 I a}{2\pi} \int_0^{\pi} \frac{\cos\phi \, d\phi}{\sqrt{r^2 + a^2 + (z-h)^2 - 2ar\cos\phi}} \hat{\boldsymbol{\phi}}. \tag{12}$$

We will now rewrite (12) so that it can be evaluated with elliptic integrals. The denominator inside the integral can be rewritten as

$$\sqrt{(r+a)^2 + (z-h)^2 - 2ar(1+\cos\phi)} = \sqrt{(r+a)^2 + z^2 - 4ar\left(\frac{1+\cos\phi}{2}\right)}. \tag{13}$$

If we let $\phi = \pi - 2\alpha$ then

$$\cos\phi = -\cos 2\alpha.$$

Since $\cos 2\alpha = 1 - 2\sin^2\alpha$,

$$\left(\frac{1+\cos\phi}{2}\right) = \sin^2\alpha.$$

With this substitution and factoring out $(r+a)^2 + (z-h)^2$ (13) becomes

$$\sqrt{(r+a)^2 + (z-h)^2}\sqrt{1 - \frac{4ar}{(r+a)^2 + (z-h)^2}\sin^2\alpha}.$$

If we define k as

$$k = \sqrt{\frac{4ar}{(r+a)^2 + (z-h)^2}} \tag{14}$$

then the denominator becomes

$$\frac{2\sqrt{ar}}{k}\sqrt{1 - k^2\sin^2\alpha}. \tag{15}$$

The numerator inside the integral in (12) can be written as

$$-(1 - 2\sin^2\alpha). \tag{16}$$

By (15), (16), and the fact that $d\phi = -2\, d\alpha$, (12) becomes

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0 I a}{2\pi} \int_{\frac{\pi}{2}}^{0} \frac{2(1 - 2\sin^2\alpha)\, d\alpha}{\frac{2\sqrt{ar}}{k}\sqrt{1 - k^2\sin^2\alpha}} \hat{\boldsymbol{\phi}} \\ &= \frac{\mu_0 I k}{2\pi}\sqrt{\frac{a}{r}}\left[\int_0^{\frac{\pi}{2}} \frac{-d\alpha}{\sqrt{1 - k^2\sin^2\alpha}} + \int_0^{\frac{\pi}{2}} \frac{2\sin^2\alpha\, d\alpha}{\sqrt{1 - k^2\sin^2\alpha}}\right] \hat{\boldsymbol{\phi}} \end{aligned} \tag{17}$$

By (1) The first term in (17) is $K(k)$. For the second term we note that

$$\frac{\sin^2\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \frac{1}{k^2}\left(\frac{1}{\sqrt{1-k^2\sin^2\alpha}} - \sqrt{1-k^2\sin^2\alpha}\right)$$

and that therefore the second term becomes

$$\frac{2}{k^2}\left[\int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} - \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\alpha}\, d\alpha\right]$$

which by (1) and (2) becomes

$$\frac{2}{k^2}(K(k)-E(k)).$$

Therefore (17) becomes

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0 I k}{2\pi}\sqrt{\frac{a}{r}}\left[\frac{2}{k^2}(K(k)-E(k)) - K(k)\right]\hat{\boldsymbol{\phi}} \\ &= \frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\left(\frac{2}{k}-k\right)K(k) - \frac{2}{k}E(k)\right]\hat{\boldsymbol{\phi}}. \qquad (18) \end{aligned}$$

2.2 Magnetic Field

In this section we take the curl of (18) to obtain $\mathbf{B}$. For a general vector function in cylindrical coordinates

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}} + v_z\hat{\mathbf{z}}$$

the curl is given by

$$\nabla\times\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial}{\partial r}r v_\phi - \frac{\partial v_r}{\partial\phi}\right)\hat{\mathbf{z}}.$$

Since $\mathbf{A}$ only has a ϕ component

$$\nabla\times\mathbf{A} = -\frac{\partial A}{\partial z}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial r}rA\,\hat{\mathbf{z}}. \qquad (19)$$

We now find the derivatives of k with respect to z and r. From (14) we see that

$$\begin{aligned} \frac{\partial k}{\partial z} &= \frac{1}{2}\sqrt{\frac{(r+a)^2+(z-h)^2}{4ar}}\cdot\frac{-8ar(z-h)}{\left[(r+a)^2+(z-h)^2\right]^2} \\ &= -\frac{4ar(z-h)}{\left[(r+a)^2+(z-h)^2\right]^2 k} = -\frac{k^2(z-h)}{\left[(r+a)^2+(z-h)^2\right]k} \\ &= -\frac{k^3(z-h)}{4ar} \qquad (20) \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial k}{\partial r} &= \frac{1}{2}\sqrt{\frac{(r+a)^2+(z-h)^2}{4ar}} \cdot \frac{4a\left[(r+a)^2+(z-h)^2\right]-8ar(z-h)}{\left[(r+a)^2+(z-h)^2\right]^2} \\
&= \frac{2a}{\left[(r+a)^2+(z-h)^2\right]^2 k} - \frac{4ar(r+a)}{\left[(r+a)^2+(z-h)^2\right]^2 k} \\
&= \frac{2ak}{4ar} - \frac{k^2(r+a)}{\left[(r+a)^2+(z-h)^2\right] k} = \frac{k}{2r} - \frac{k^3(r+a)}{4ar}. \qquad (21)
\end{aligned}$$

We will now find the r component of $\mathbf{B}$. From (18) we see that

$$-\frac{\partial A}{\partial z} = -\frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\left(-\frac{2}{k^2}-1\right)\frac{\partial k}{\partial z}K + \left(\frac{2}{k}-k\right)\frac{dK}{dk}\frac{\partial k}{\partial z} + \frac{2}{k^2}\frac{\partial k}{\partial z}E - \frac{2}{k}\frac{dE}{dk}\frac{\partial k}{\partial z}\right].$$

Substituting the values from (6), (8), and (20) we have

$$-\frac{\partial A}{\partial z} = -\frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\Bigg[\left(\frac{2}{k^2}+1\right)\frac{k^3(z-h)}{4ar}K - \left(\frac{2}{k}-k\right)\left(\frac{E}{k(1-k^2)}-\frac{K}{k}\right)\frac{k^3(z-h)}{4ar} - \frac{2}{k^2}\frac{k^3(z-h)}{4ar}E + \frac{2}{k}\left(\frac{E}{k}-\frac{K}{k}\right)\frac{k^3(z-h)}{4ar}\Bigg].$$

Expanding we have

$$-\frac{\partial A}{\partial z} = -\frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\Bigg[\left(\frac{k(z-h)}{2ar}+\frac{k^3(z-h)}{4ar}\right)K - \frac{k(z-h)E}{2ar(1-k^2)} + \frac{k(z-h)K}{2ar} + \frac{k^3(z-h)E}{4ar(1-z^2)} - \frac{k^3(z-h)K}{4ar} - \frac{k(z-h)E}{2ar} + \frac{k(z-h)E}{2ar} - \frac{k(z-h)K}{2ar}\Bigg]$$

which simplifies to

$$\begin{aligned}
-\frac{\partial A}{\partial z} &= -\frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\frac{k(z-h)}{2ar}K + \frac{-2k(z-h)+k^3(z-h)}{4ar(1-k^2)}E\right] \\
&= -\frac{\mu_0 I k(z-h)}{4\pi\sqrt{ar^3}}\left[K - \frac{2-k^2}{2(1-k^2)}E\right]. \qquad (22)
\end{aligned}$$

We now find the z component of $\mathbf{B}$. From (18) we see that

$$rA = \frac{\mu_0 I\sqrt{ar}}{2\pi}\left[\left(\frac{2}{k}-k\right)K(k) - \frac{2}{k}E(k)\right]$$

and that, therefore,

$$\frac{\partial}{\partial r}rA = \frac{\mu_0 I a}{4\pi\sqrt{ar}}\left[\left(\frac{2}{k}-k\right)K-\frac{2}{k}\right]+\frac{\mu_0 I\sqrt{ar}}{2\pi}\left[\left(-\frac{2}{k^2}-1\right)\frac{\partial k}{\partial r}K + \left(\frac{2}{k}-k\right)\frac{dK}{dk}\frac{\partial k}{\partial r}+\frac{2}{k^2}\frac{\partial k}{\partial r}E-\frac{2}{k}\frac{dE}{dk}\frac{\partial k}{\partial r}\right]. \quad (23)$$

Substituting the values from (6), (8), and (21) we have

$$\frac{\partial}{\partial r}rA = \frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\left(\frac{1}{k}-\frac{k}{2}\right)K-\frac{E}{k}+\left(-\frac{2r}{k^2}-r\right)\left(\frac{k}{2r}-\frac{k^3(r+a)}{4ar}\right)K + \left(\frac{2r}{k}-kr\right)\left(\frac{E}{k(1-k^2)}-\frac{K}{k}\right)\left(\frac{k}{2r}-\frac{k^3(r+a)}{4ar}\right) + \frac{2r}{k^2}\left(\frac{k}{2r}-\frac{k^3(r+a)}{4ar}\right)E-\frac{2r}{k}\left(\frac{E}{k}-\frac{K}{k}\right)\left(\frac{k}{2r}-\frac{k^3(r+a)}{4ar}\right)\right]. \quad (24)$$

Expanding we have

$$\frac{\partial}{\partial r}rA = \frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\left(\frac{1}{k}-\frac{k}{2}\right)K-\frac{E}{k}+\left(-\frac{1}{k}+\frac{k(r+a)}{2a}-\frac{k}{2}+\frac{k^3(r+a)}{4a}\right)K+ \frac{E}{k(1-k^2)}-\frac{K}{k}-\frac{kE}{2(1-k^2)}+\frac{kK}{2}-\frac{k(r+a)E}{2a(1-k^2)}+\frac{k(r+a)K}{2a}+\frac{k^3(r+a)E}{4a(1-k^2)}- \frac{k^3(r+a)K}{4a}+\left(\frac{1}{k}-\frac{k(r+a)}{2a}\right)E-\frac{E}{k}+\frac{k(r+a)E}{2a}+\frac{K}{k}-\frac{k(r+a)K}{2a}\right]. \quad (25)$$

This simplifies to

$$\frac{\partial}{\partial r}rA = \frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\left(-\frac{k}{2}+\frac{k(r+a)}{2a}\right)K+\left(\frac{1}{k(1-k^2)}-\frac{k}{2(1-k^2)}- \frac{k(r+a)}{2a(1-k^2)}+\frac{k^3(r+a)}{4a(1-k^2)}-\frac{1}{k}\right)E\right]. \quad (26)$$

Dividing by r to get the z component of $\mathbf{B}$ we have

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}rA &= \frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r^3}}\left[\frac{k(r+a)-ak}{2a}K+\frac{k^2\left(-2a-2(r+a)+k^2(r+a)+4a\right)}{4ak(1-k^2)}E\right] \\ &= \frac{\mu_0 Ikr}{4\pi\sqrt{ar^3}}\left(K+\frac{k^2(r+a)-2r}{2r(1-k^2)}E\right). \end{aligned} \quad (27)$$

Since $\mathbf{B} = \nabla \times \mathbf{A}$ by (19), (22), and (27)

$$\mathbf{B}(r,z) = \frac{\mu_0 I k}{4\pi\sqrt{ar^3}}\left[-(z-h)\left(K - \frac{2-k^2}{2(1-k^2)}E\right)\hat{\mathbf{r}} + r\left(K + \frac{k^2(r+a)-2r}{2r(1-k^2)}E\right)\hat{\mathbf{z}}\right]. \tag{28}$$

It is easy to calculate the magnetic field due to the current loop at a point located on the z axis using the Biot-Savart law

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{s}\times\hat{\mathbf{R}}}{R^2}.$$

In this case $d\mathbf{s}$ is perpendicular to $\hat{\mathbf{R}}$ so that $d\mathbf{s}\times\hat{\mathbf{R}} = ds$. For every line element $d\mathbf{s}$ the radial component of $d\mathbf{B}$ cancels with the radial component due to the line element on the opposite side of the loop. Therefore $\mathbf{B}$ is only in the z direction and we have

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{\cos\theta}{R^2}\int ds\,\hat{\mathbf{z}}$$

where θ is the angle made by R and a. $\cos\theta = \frac{a}{R}$ and $\int ds = 2\pi a$ so that

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{a}{R^3}2\pi a\,\hat{\mathbf{z}}.$$

since

$$R = \sqrt{a^2 + (z-h)^2}$$

we have

$$\mathbf{B} = \frac{\mu_0 I a^2}{2\left[a^2 + (z-h)^2\right]^{\frac{3}{2}}}\hat{\mathbf{z}}. \tag{29}$$

We now show that the magnetic field for a point along the z axis calculated in (29) is equivalent to the magnetic field calculated in (28) for $r = 0$. We will first show that $B_r = 0$ when $r = 0$ and then show that B_z is equal to (29). First we note from (14) that $k = 0$ when $r = 0$. Now we rewrite B_r given by (22) to eliminate the r from the denominator

$$\begin{aligned} B_r &= -\frac{\mu_0 I(z-h)}{2\pi r\sqrt{(r+a)^2+(z-h)^2}}\left[K - \frac{E}{1-k^2} + \frac{k^2 E}{2(1-k^2)}\right] \\ &= -\frac{\mu_0 I(z-h)}{2\pi\sqrt{(r+a)^2+(z-h)^2}}\left[\frac{K}{r} - \frac{E}{r(1-k^2)} + \frac{2aE}{\left[(r+a)^2+(z-h)^2\right](1-k^2)}\right] \end{aligned} \tag{30}$$

Expressing the first two terms inside the braket in (30) with the definitions of K and E given in (1) and (2) we have

$$\begin{aligned}\frac{K}{r}-\frac{E}{r(1-k^2)} &= \int_0^{\frac{\pi}{2}}\frac{d\alpha}{r\sqrt{1-k^2\sin^2\alpha}}-\int_0^{\frac{\pi}{2}}\frac{\sqrt{1-k^2\sin^2\alpha}}{r(1-k^2)}d\alpha \\ &= \frac{1}{r}\int_0^{\frac{\pi}{2}}\frac{k^2\sin^2\alpha-k^2}{(1-k^2)\sqrt{1-k^2\sin^2\alpha}}d\alpha \\ &= \frac{4a}{(r+a)^2+(z-h)^2}\int_0^{\frac{\pi}{2}}\frac{\sin^2\alpha-1}{(1-k2)\sqrt{1-k^2\sin^2\alpha}}d\alpha.\end{aligned}$$

Substituting $r=0$ for the first two terms we have

$$\begin{aligned}\frac{4a}{a^2+(z-h)^2}\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha-1\right)d\alpha &= \frac{4a}{a^2+(z-h)^2}\left[\left(\frac{\alpha}{2}-\frac{1}{4}\sin 2\alpha\right)\Big|_0^{\frac{\pi}{2}}-\frac{\pi}{2}\right] \\ &= \frac{4a}{a^2+(z-h)^2}\left[\left(\frac{\pi}{4}-0\right)-\frac{\pi}{2}\right] \\ &= -\frac{\pi a}{a^2+(z-h)^2}. \qquad (31)\end{aligned}$$

Letting $r=0$, substituting (31) into (30), and from (3) we have

$$B_r=-\frac{\mu_0 I(z-h)}{2\pi\sqrt{a^2+(z-h)^2}}\left[-\frac{\pi a}{a^2+(z-h)^2}+\frac{\pi a}{a^2+(z-h)^2}\right]=0.$$

We now rewrite B_z given by (27) to eliminate the r from the denominator to get

$$\begin{aligned}B_z &= \frac{\mu_0 I}{2\pi r\sqrt{(r+a)^2+(z-h)^2}}\left[rK+\left(\frac{k^2(r+a)}{2(1-k^2)}-\frac{r}{1-k^2}\right)E\right] \\ &= \frac{\mu_0 I}{2\pi\sqrt{(r+a)^2+(z-h)^2}}\left[K(k)+\left(\frac{2a(r+a)}{[(r+a)^2+(z-h)^2](1-k^2)}-\frac{1}{1-k^2}\right)E(k)\right].\end{aligned}$$

Letting $r=0$ we have from (3)

$$\begin{aligned}B_z &= \frac{\mu_0 I}{2\pi\sqrt{a^2+(z-h)^2}}\frac{\pi}{2}\left[1+\frac{2a^2}{a^2+(z-h)^2}-1\right] \\ &= \frac{\mu_0 I a^2}{2\left[a^2+(z-h)^2\right]^{\frac{3}{2}}}.\end{aligned}$$

Since $B_r=0$,

$$\mathbf{B}=\frac{\mu_0 I a^2}{2\left[a^2+(z-h)^2\right]^{\frac{3}{2}}}\hat{\mathbf{z}}$$

which is the same as (29).

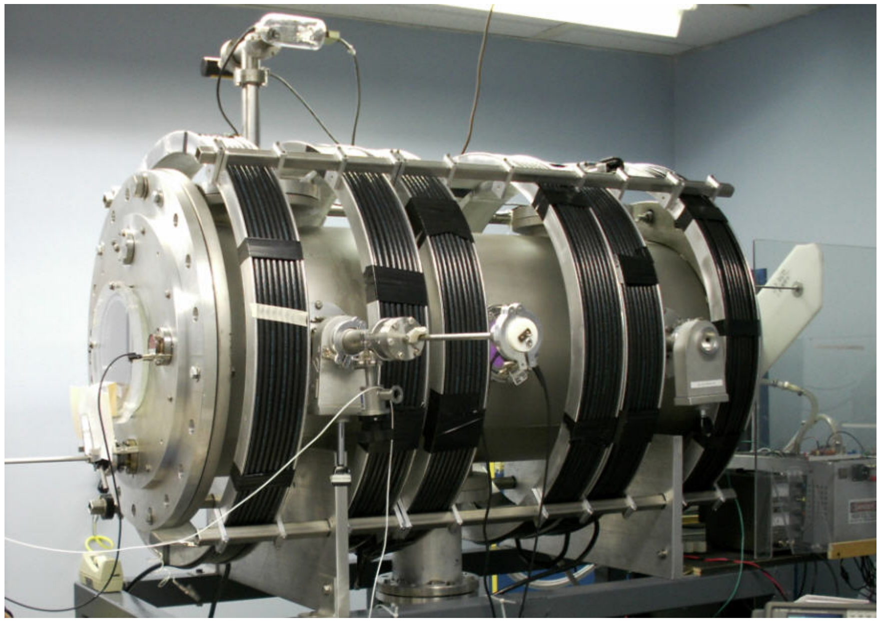

Figure 1: UCLA high school plasma machine. The coil on the far right is disconected because it interfered with the plasma source.

3 Magnetic Field Inside Plasma Chamber

In this section we extend the solution for one current loop (28) to the plasma chamber in the UCLA high school lab. The magnetic field of the plasma machine is made up of five groups of coils of wire that surround the chamber, see Fig. 1. Each group has a total of 24 turns (eight turns long and three turns deep). We will approximate each turn to be one current loop and add the contributions from each of the 120 loops.

We set up a cylindrical coordinate system with the origin located at the end of the chamber opposite the plasma source and the z axis along the axis of the chamber. We take the radius of each loop of wire to be the distance from the center of the chamber to the center of the 1.2 cm diameter wire. We let a_0 be the radius of the 40 innermost wires and Δa be the distance between the centers of two touching wires. The radius of the 40 middle wires is, therefore, $a_0 + \Delta a$ and

Figure 2: Magnetic field inside the plasma chamber.

the radius of the 40 outermost wires is $a_0 + 2\Delta a$. $a_0 = 43.4$ cm and Δa is the diameter of a wire and is therefore 1.2 cm.

We let h_l be the distance along the z axis from the origin to the centers of the first three wires in the lth group of wires. Thus, the distance along the z axis from the origin to the centers of the second three wires in the lth group of wires is given by $h_l + \Delta a$ and to the last three wires by $h_l + 7\Delta a$. For the plasma machine the h_l are (from $l = 1$ to $l = 5$) 8.3 cm, 31.2 cm, 43.4 cm, 74 cm, and 84.9 cm.

The magnetic field at one point in the plasma chamber is the sum of the magnetic field due to each of the 120 current loops. We rewrite the $\mathbf{B}(r, z)$ in (28) to $\mathbf{B}_L(r, z, a, h)$ to indicate that the field given by (28) is the field at a point with coordinates (r, ϕ, z), for any ϕ, due to a single current loop with radius a and distance along the z axis h. The total magnetic field at a point (r, ϕ, z) is given by

$$\mathbf{B}(r, z) = \sum_{l=1}^{5} \sum_{j=0}^{7} \sum_{i=0}^{2} \mathbf{B}_L(r, z, a_0 + i\Delta a, h_l + j\Delta a). \qquad (32)$$

Fig. 2 shows the magnetic field inside the plasma chamber as calculated by (32).

References

[1] George B. Arfken and Hans J. Weber. *Mathematical Methods for Physicists*. Elsevier Academic Press, New York, 6 edition, 2005.

[2] Albert Shadowitz. *The Electromagnetic Field*. Dover Publications, Inc., New York, 1975.