Laptag lectures on resonance cones and what you need to derive them

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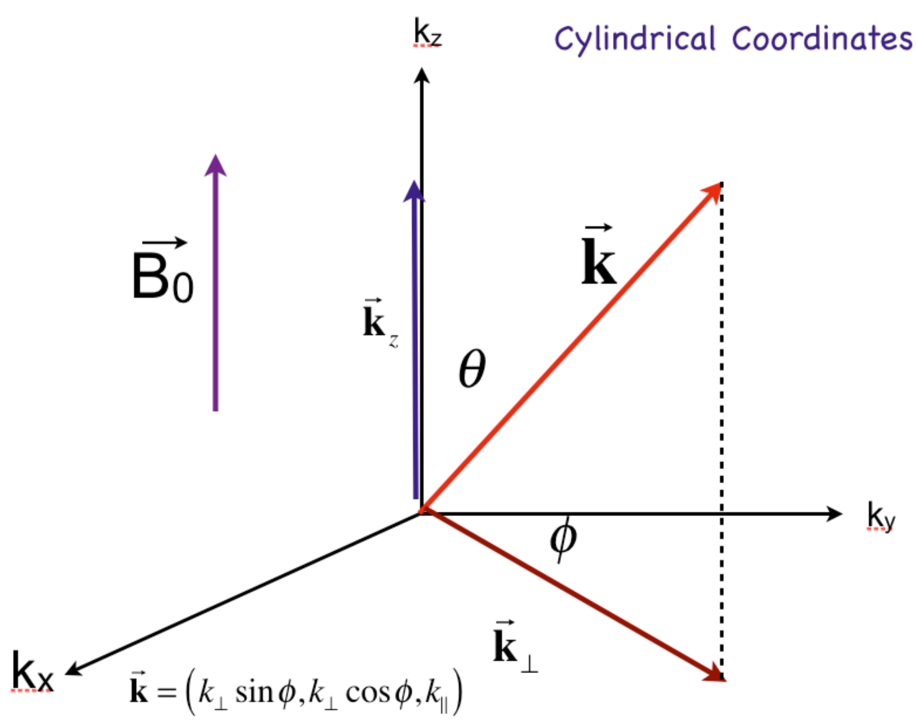
(IF students, teachers contribute to this their names will be added!)

The object of this is to derive the formula for resonance cones in a plasma.   There are a number of steps along the way. We have to discuss:

a) What is the index of refraction of a plasma?
b) Wave propagation and - What type of plasma waves are we talking about?
c) What is a dielectric in free space and in a plasma?
d) Coordinate systems
e) Electric potential and electric fields
f) Fourier analysis (This w is done in detail in other LAPTAG notes on the website, download them and read them NOW!)
g) Vectors and complex numbers
h) Tensors and matrices
First of all coordinates systems. The first coordinate system you learn about is rectangular coordinates. There a vector r is given by its x,y and z coordinate positions.



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = (x,y,z)$. Here $\hat{i},\hat{j},\hat{k}$ are vectors of length 1 (in any coordinate system, in mks they are 1 meter long, in the English system they are 1 inch long, etc). The vector may be drawn as:

Figure 1

We can also express the vector r in spherical coordinates. Here the vector is determined not by how long its projection is on the three axis but by its length in the $\hat{r}$ direction (this is a pointer from the origin towards the tip of the vector, it also has length 1). The vector is described not by 3 lengths but by one length and two angles $\vec{r} = (r,\theta,\phi)$ shown in figure 1.

To go from spherical coordinates to rectangular the transformation is:

$x = r\cos\theta\sin\phi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\phi$

Problem : Find the inverse transformation. That is what is r,θ,ϕ if you are given x,y,z

Electric Force. The force on one charge due to another is:
Note we are in the mks system of units

(1) $\vec{F} = \dfrac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{r}$ It is a vector quantity .

This is the force on Q due to the presence of charge q. The direction is on the straight line joining them. This is of the same form as gravitation (accident?) It is attractive or repulsive depending on the product of qQ.

What if you have 3 charges? Then we have to decide which charge we wish to calculate the force on. For example

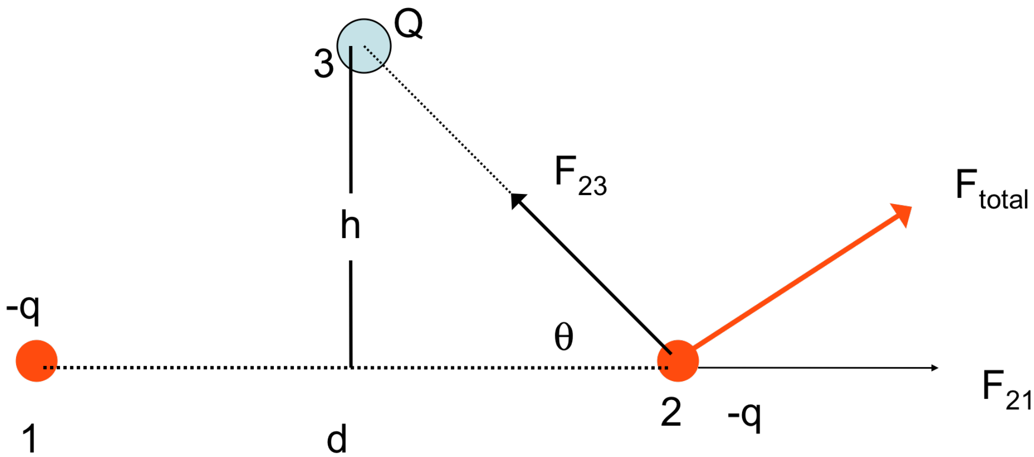


Figure 2- 3 charges
The two "red" charges are positive and the "blue one is negative. What are the forces on charge 2 on the right? The negative charges repel and F_{21} is the force on 2 due to charge 1. F_{23} as shown is attractive. To calculate them we simply use the force law:

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0} q^2 \frac{1}{d^2} \hat{x}$$

$$\vec{F}_{23} = \frac{-1}{4\pi\varepsilon_0} qQ \frac{1}{\left(\frac{d}{2}\right)^2 + h^2} \cos\theta\, \hat{x} + \frac{1}{4\pi\varepsilon_0} qQ \frac{1}{\left(\frac{d}{2}\right)^2 + h^2} \sin\theta\, \hat{y}$$

$$\cos\theta = \frac{d/2}{\sqrt{\left(\frac{d}{2}\right)^2 + h^2}} \qquad F_{total} = \vec{F}_{23} + \vec{F}_{21}$$

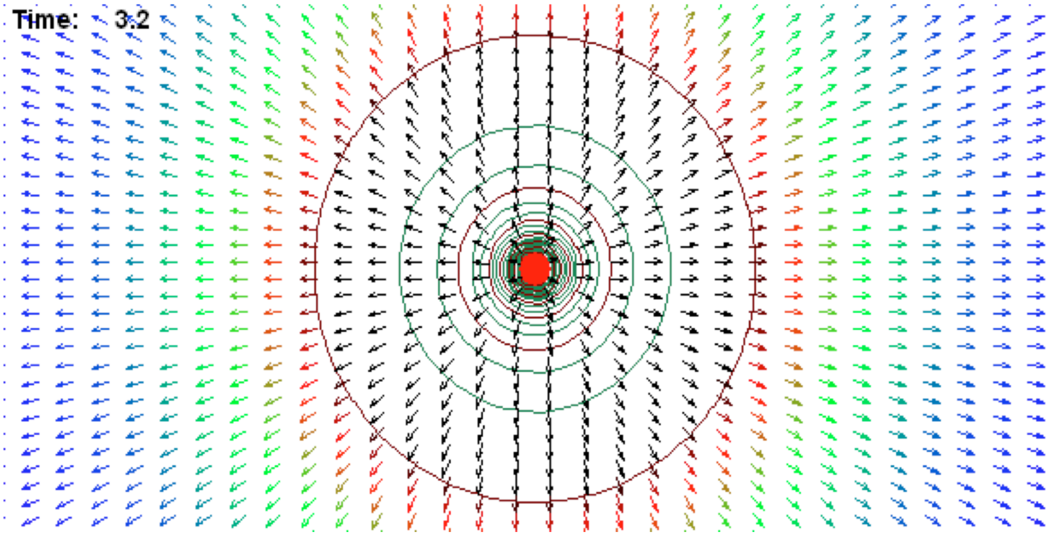
Suppose our test charge q is exposed to more than one charge. The electric field (or force it experiences) is the vector sum of the forces from all the charges.

There may be discrete charges, line charges surface charges of volume filled charges.

The electric field (also a vector is defined as the force on an infinitesimally small charge:

(2) $$\vec{\mathbf{E}} = \vec{\mathbf{F}} / q = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}} \qquad \text{Lim } q \to 0$$

Which is real **F** or **E** ? The utility of the electric field is that all that occurs in it is the source charge Q. The test charge serves as a point indicator telling us how big the field is and which way it points. The next figure (Figure 3) is the electric field due to a single point charge:



The charge (red) is at the center. It must be positive because the field lines point outwards. The vectors shown are unit vectors. They all have the same size to make the picture clear. Actually they get smaller by $\frac{1}{r^2}$ as you go outwards.

If there are more than one charge than

$$Q_{tot} = \sum_i Q_i \quad \text{sum of discrete charges}$$

$$Q_{tot} = \int \lambda dl \quad \text{line charge } \lambda = \frac{Q}{L}$$

$$Q_{tot} = \int \sigma dA \quad \text{surface charge } \sigma = \frac{Q}{A}$$

$$Q_{tot} = \int \rho dV \quad \text{volume charge } \rho = \frac{Q}{V}$$

For the case of a uniform charge density , $dq = \rho dV$ here small volume element contain's a small amount of charge. Then we can re-write (2) as

(3) $\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\vec{r}}{r^2}dV$ This is how to calculate the the electric field due to a charge density filling a volume V.

In the early 1800's Carl Fredrich Gauss (shown above) through experimentation discovered that :

(4) $\oint \vec{\mathbf{E}} \bullet \hat{n}\, dA = \frac{q_{enclosed}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho\, dV$ The term on the left is the total charge divided by ε_0 and the formal name for the term on the left is the electric flux. Consider a closed surface such as the one in figure 1

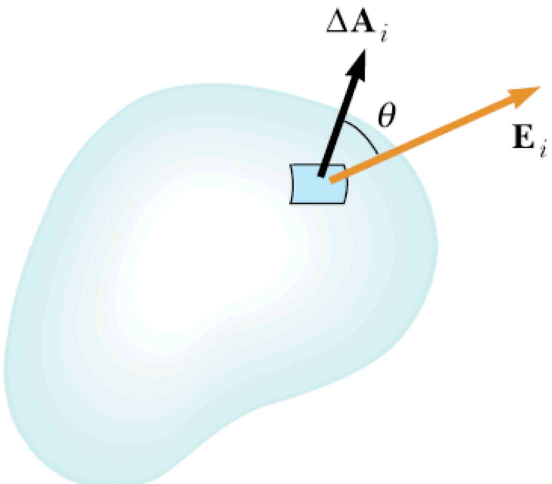


Figure 4 closed Gaussian surface this bounds a volume, think of it as the skin stretched over the volume. dA is a small patch of surface area and the black arrow is the normal vector to the surface (perpendicular to the surface).

$\nabla\Phi_i = E_i \Delta A_i \cos\theta_i = \vec{E}_i \bullet \hat{n} dA_i$. This is an element of electric flux through the surface. The definition of electric flux is

(5) $\Phi_E = \oint \vec{\mathbf{E}} \bullet \hat{n}\, dA$

There are other types of flux (magnetic flux , particle flux …) . The integral tells us the total flux crossing the "blue" surface.

Consider this rectangular patch. In figure 4

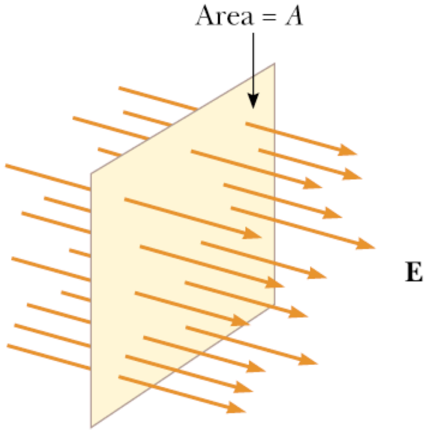


Figure 5 Electric field lines crossing a square area. The electric flux is proportional to the number of lines that cross.

As an example. Suppose we chose our area as the surface of a closed cube and there is an electric field constant in space as shown below in figure 5:

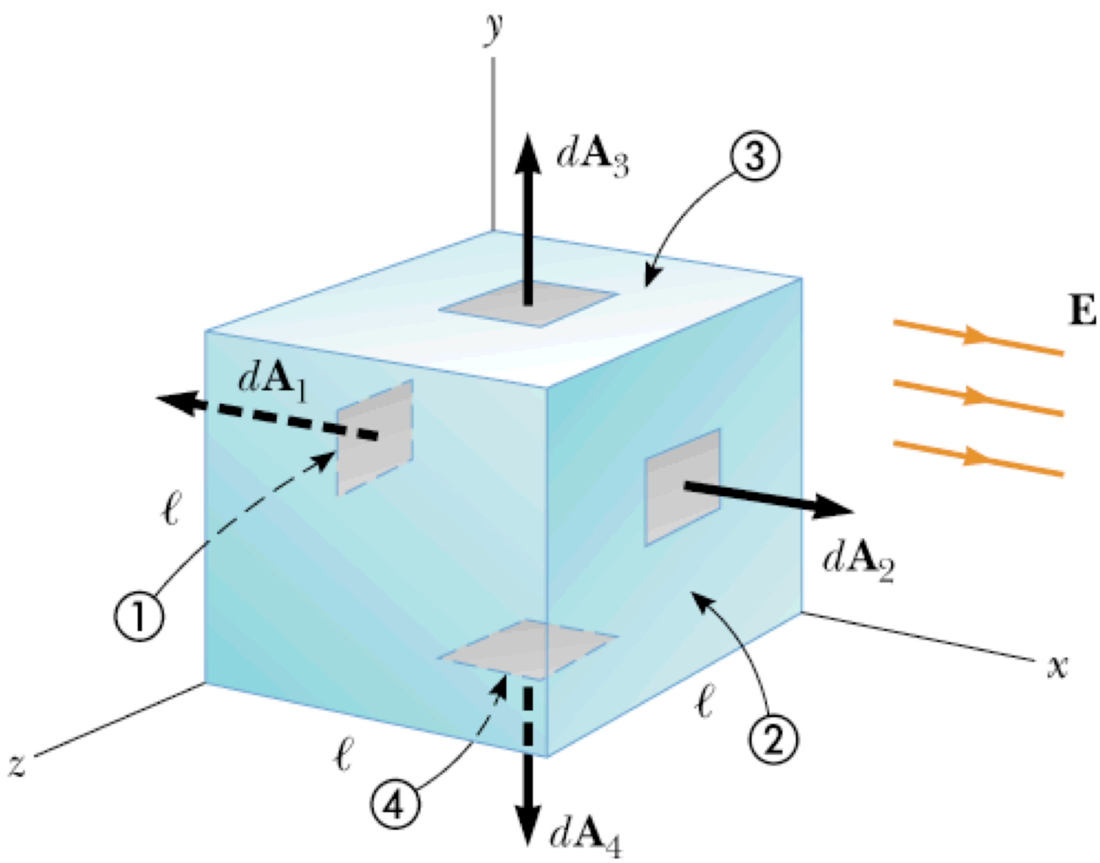
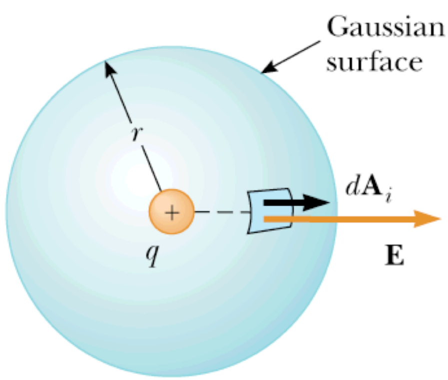


Figure 5. dA_1 and dA_2 point in opposite directions but the field points only to the right and is constant therefore $EdA_2 - EdA_1 = 0$. You can do this face by face and see that the integral is zero therefore there must be no charge in the cube.

Now let us do the same problem of a sphere of charge.



Now the patch is a patch of the surface and:

$$\phi_E = \int_{surface} \vec{\mathbf{E}} \bullet \hat{n}\, dA = \frac{q_{enclosed}}{\varepsilon_0} = \int_{surface} \vec{\mathbf{E}} \bullet \hat{r}\, dA = \int E\, dA = E \int dA$$

$$\phi_E = E 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$E = \frac{q}{4\pi r^2 \varepsilon_0}$$

Note that is q was negative the answer would be negative implying the field lines point inwards.

We have Gauss's law as in integral. To derive our plasma equations we need it in differential form. We do this using a famous theorem called the divergence theorem:

(6) $\int_{vol} \nabla \bullet \vec{A}\, dV = \int_{closed-area} \nabla \bullet \hat{n} dA$ Where A is any vector. Allying this to equation (4)

$\oint \vec{\mathbf{E}} \bullet \hat{n}\, dA = \int \nabla \bullet \vec{E}\, dV = \frac{1}{\varepsilon_0} \int \rho\, dV$. Since both integrals are over the same volume whats inside of them (the integrand) must be equal.

(7) $\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$ This is the differential form of Gauss's law.

This equation is derived for media where the dielectric constant ε_0 which reflects the property of the media is just a number. This constant relates charge to electric field. In a plasma with a magnetic field this is no longer a constant. It becomes a mathematical function which depends upon the plasma density, magnetic field and so on. Because of the magnetic it has different values in different directions. It could be a vector but it is not because if you multiply two vectors you get 9 components. It must be a tensor. We will have more to say about tensors as we go on. But now we must re-write (7) as

(8) $\nabla \bullet \left(\vec{\varepsilon} \vec{E} \right) = \rho_{ext}$ Here $\varepsilon_0 \to \vec{\varepsilon}$

Coulomb's Law and $\vec{E} = -\nabla \Phi$

(9) $D = \varepsilon_0 \vec{\kappa} \bullet \vec{E}$ where we define $\vec{\kappa} = \varepsilon_0 \vec{\varepsilon}$

(10) $$\vec{\kappa} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} & 0 \\ \kappa_{yx} & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{\parallel} \end{bmatrix}$$

(11) $\kappa_{xx} = \kappa_{yy} = 1 - \frac{\omega_{pe}^2}{\left(\omega^2 - \omega_{ce}^2\right)}$

(12) $\kappa_{xy} = \kappa_{yx} = \frac{i\omega_{ce}\omega_{pe}^2}{\left(\omega^2 - \omega_{ce}^2\right)}$

(13) $\kappa_{\parallel} = 1 - \frac{\omega_{pe}^2}{\omega^2}$

(14) $\rho_{ext} = qe^{-i\omega t}\delta(\vec{r})$

Before we go to the nest step we have to introduce the concept of gradient and potential gradient

We first start with the definition of work.

(15) $W = \int_a^b \left(\vec{F}\bullet d\vec{r}\right)$; unit of work is the Joule

Consider a spring. The force to contract/stretch it is : $\vec{F} = -k\left(x - x_0\right)$.

Here x_0 is the equilibrium position. If it is compressed $x<x_0$ and the force is negative (you have to exert a force in the negative x direction. The spring is anchored to a wall located at negative x.

The work done to compress/stretch the spring is :

(16) $W = \int_a^b F_x dx \;\; = \mathrm{k}\int_0^2 \mathrm{x\,dx} = \frac{1}{2}kx^2$; note it is always positive.

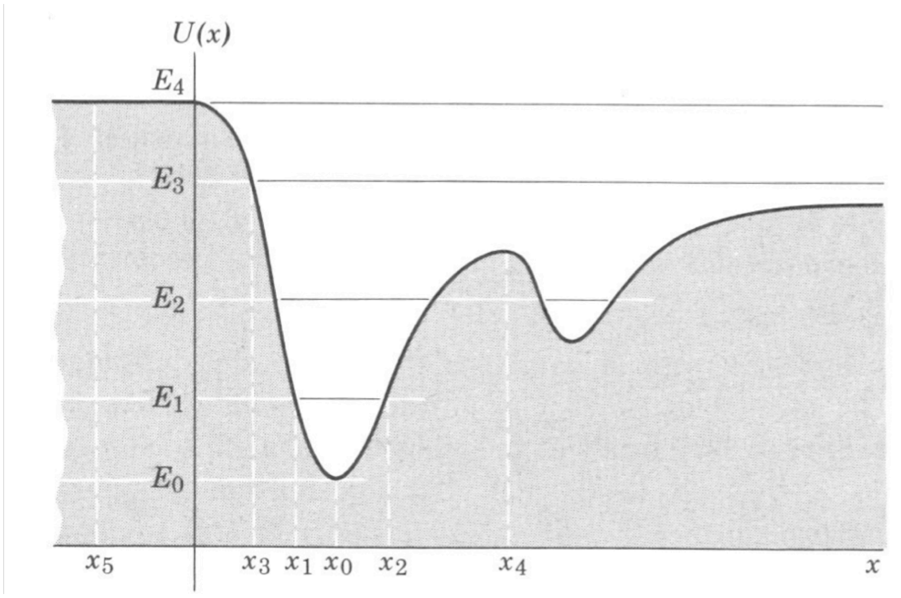
Consider again a system where U+W=E. (W is the kinetic energy, U the potential energy, E the total energy which is constant if energy is conserved). Take the differential of equation (9)

In 1D

(17) $\Delta W = F_x dx$; $\frac{\Delta U}{dx} = -F_x$

In three dimensions $\vec{F} = -\nabla U$ where $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ (This is the gradient)

Diagrams of potential are incredibly useful in predicting how a system will go.



a) Lowest possible energy is E_0 (need $E > U(x)$ for real velocity)

b) $E<E_1$ trapped symmetrically oscillating particle (stable equilibruim)

c) $E<E_1$ trapped oscillating particle

d) $E<E_3$ particle can leave to right , but can't pass x=0

e) $E<E_4$ particle keeps on going either way

Using Fourier Analysis equation (8) and (8) become

$$-\varepsilon_0 i\vec{\mathbf{k}}\cdot\vec{\kappa}\bullet\nabla\Phi = qe^{i\omega t}\delta(\vec{r}) \quad \text{or}$$

$$-\varepsilon_0 i\vec{\mathbf{k}}\cdot\vec{\kappa}\bullet i\vec{\mathbf{k}}\Phi = qe^{i\omega t}\delta(\vec{r}) = \varepsilon_0\vec{\mathbf{k}}\cdot\vec{\kappa}\bullet\vec{\mathbf{k}}\Phi$$

Now we have to do the algebra

(18) $$\varepsilon_0\vec{\mathbf{k}}\cdot\vec{\kappa}\bullet\vec{\mathbf{k}}\Phi = \varepsilon_0\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}\begin{bmatrix} \kappa_{xx} & -\kappa_{xy} & 0 \\ \kappa_{yx} & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{\parallel} \end{bmatrix}\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}\Phi$$

$$= \varepsilon_0 \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \begin{bmatrix} \kappa_{xx}k_x - \kappa_{xy}k_y & \kappa_{yx}k_x + \kappa_{yy}k_y & \kappa_{\parallel}k_z \end{bmatrix} \Phi = \varepsilon_0 \left(\kappa_{xx}k_x^2 - \kappa_{xy}k_xk_y + \kappa_{yx}k_xk_y + \kappa_{yy}k_y^2 + \kappa_{\parallel}k_z^2 \right)$$

$$= \varepsilon_0 \left(\kappa_{xx}k_x^2 + \kappa_{yy}k_y^2 + \kappa_{\parallel}k_z^2 \right) \Phi$$

$$\left(\kappa_{\perp}^2 k_{\perp}^2 + \kappa_{\parallel}k_z^2 \right) \Phi = \frac{q}{\varepsilon_0} e^{i\omega t} \delta(\vec{r})$$

(19) $$\Phi(k) = \frac{q}{\varepsilon_0 \left(\kappa_{\perp}^2 k_{\perp}^2 + \kappa_{\parallel}k_z^2 \right)} e^{-i\omega t} \delta(\vec{r})$$

Then use the definition of the 3D delta function

(20) $$\delta(\vec{r}) = \frac{1}{(2\pi)^3} \int_0^{\infty} e^{i2\pi \vec{\mathbf{k}} \cdot \vec{r}} d^3k$$

(21) $$\Phi(k) = \frac{qe^{-i\omega t}}{\varepsilon_0} \iiint \frac{e^{i\vec{\mathbf{k}} \cdot \vec{r}}}{\left(\kappa_{\perp}^2 k_{\perp}^2 + \kappa_{\parallel}k_z^2 \right)} e^{-i\omega t} \frac{d^3k}{(2\pi)^3}$$

We have used used the definition of the delta function
There are many alternate approximate definitions of the delta function some are:

$$\delta(x) = \frac{1}{2L} + \frac{1}{L} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L} \right)$$

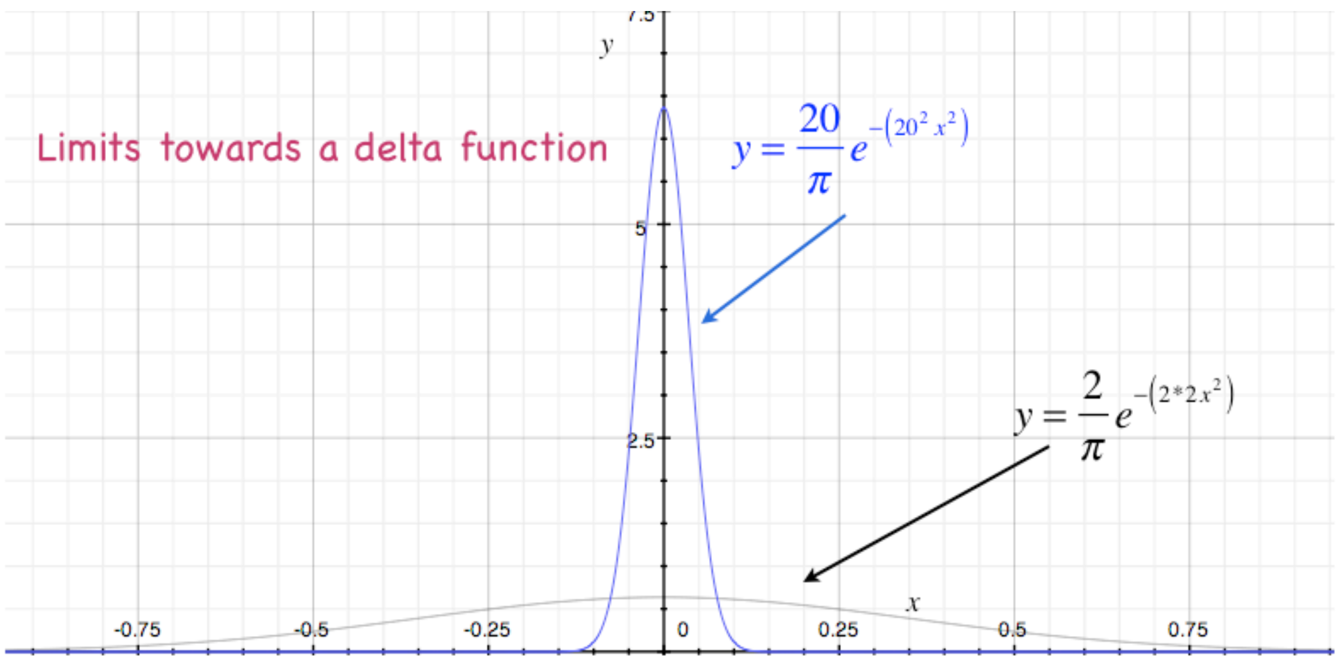
$$\delta(x) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L} \right) \cos\left(\frac{n\pi \varsigma}{L} \right) \quad , \quad 0 \le \varsigma \le \mathrm{L}$$

$$\delta(x) = \frac{\sin kx}{\pi k} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk \quad ; \quad \text{Fourier Transform}$$

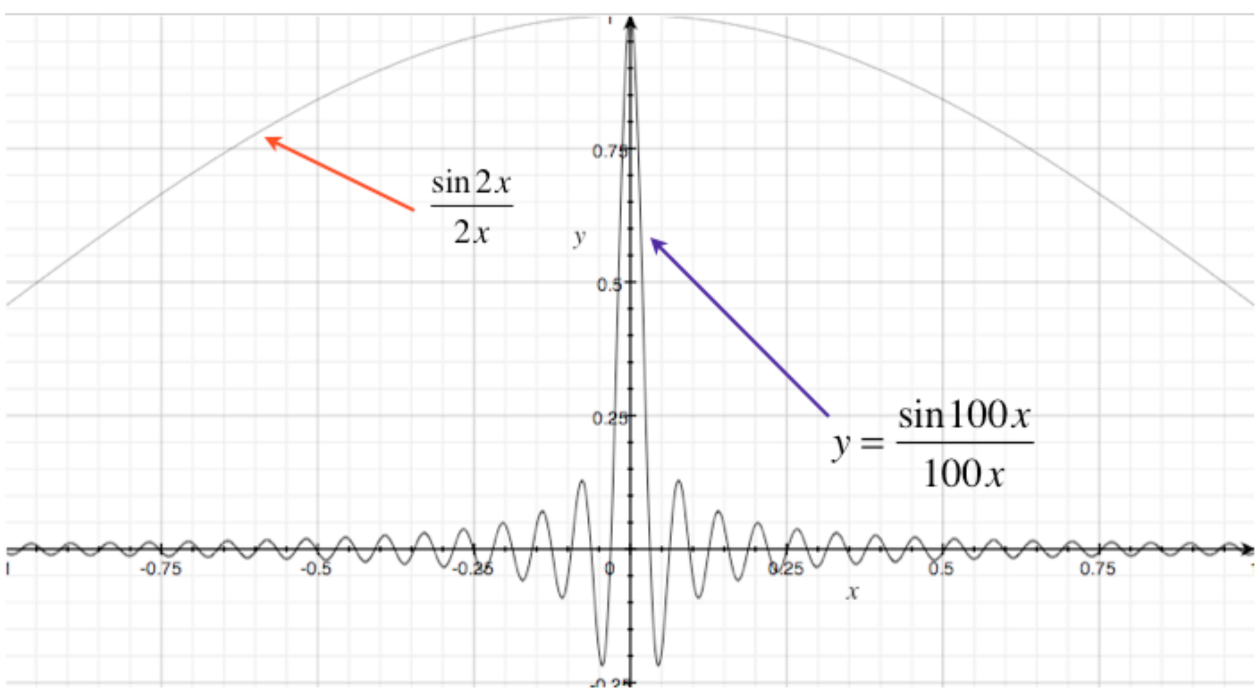
$$\delta(x) = \frac{k}{\pi} e^{-k^2x^2}$$

$$\delta(x) = \frac{k}{\pi} \frac{1}{1 + k^2\pi^2}$$

A graph of one the cases:



Another example is sin(ax)/ax shown below:

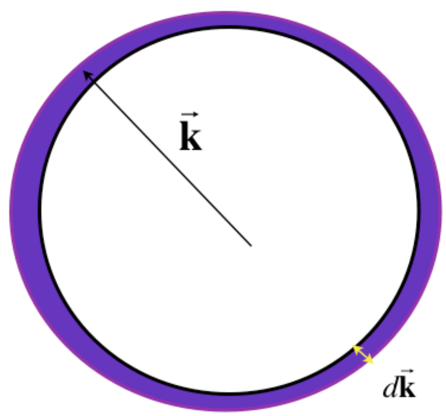


But $\vec{\mathbf{k}} = \left(k_\perp \sin\phi, k_\perp \cos\phi, k_\parallel\right)$, $\vec{r} = \left(\rho, 0, z\right)$ (It has components in xy plane and z plane only).

Therefore $\vec{\mathbf{k}} \bullet \vec{r} = \rho k_\perp \sin\phi + z k_\parallel$

$d^3k = k_\perp dk_\perp dk_\parallel d\phi$

For example for an area of element in k space



This is a volume element in cylindrical coordinates. But it is in **k** space. Why. A cylinder volume is an area times a length. The area is a circle. In r space (rather than k space)
$V = \pi r^2 z$. Here r is perpendicular to z. The differential is the differential of the area time the differential in z
$dV = d(\pi r^2)dz = 2\pi r dr dz$. In k space $k_\perp$ is the component of the wavenumber in the plane perpendicular to the magnetic field so its just like a radius. $2\pi = \int_0^{2\pi} d\phi$. If nothing depends on angle then you get 2π but if things are angularly dependent then you have to do the integral. As you see the latter is the case.

and equation 21 becomes

(22) $$\Phi(\rho,z) = \frac{qe^{i\omega t}}{\varepsilon_0} \iiint \frac{e^{i(\rho k_\perp \sin\phi + z k_\parallel)}}{\left(\kappa_\perp^2 k_\perp^2 + \kappa_\parallel k_z^2\right)} \frac{k_\perp dk_\perp d\phi dz}{(2\pi)^3} = \frac{qe^{i\omega t}}{(2\pi)^2 \varepsilon_0} \int_{k_\perp = 0}^{\infty} e^{ik_\parallel z} \int_{k_\parallel = -\infty}^{\infty} \frac{k_\perp dk_\perp dk_\parallel}{\left(\kappa_\perp^2 k_\perp^2 + \kappa_\parallel k_z^2\right)} \int_0^{2\pi} e^{i\rho k_\perp \sin\phi} \frac{d\phi}{2\pi}$$

We are integrating over all possible wavenumbers both parallel and perpendicular to B and all angles so the answer depend on space and time only.

It turns out that the integral over phi can be done! Its complicated and leads to

$$\int_0^{2\pi} e^{-i(n\beta - x\sin\beta)} d\beta = 2\pi J_n(x)$$ J is a Bessel function

Let n=0; $2\pi \int_0^{2\pi} e^{i(x\sin\beta)} d\beta = J_0(x)$ Let $x = k_\perp \rho$ Then

(23) $$\Phi(k) = \frac{qe^{-i\omega t}}{(2\pi)^2 \varepsilon_0} \int_{k_{||}=0}^{\infty} e^{ik_{||}z} \int_{k_\perp=-\infty}^{\infty} \frac{k_\perp dk_\perp dk_{||}}{\left(\kappa_\perp^2 k_\perp^2 + \kappa_{||} k_z^2\right)} J_o(k_\perp \rho)$$

What are Bessel Functions? They are in a way like sines and cosines.

The differential equation and solution for sines and cosines is

(24) $$\frac{\partial^2 y}{\partial x^2} = -\alpha^2 y \quad ; \quad \text{y=A}\cos(\alpha x) + B\sin(\alpha x)$$

Bessel functions also are solutions to (of course) Bessel's equation which is

(25) $$x^2 \frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} + \left(x^2 - \alpha^2\right) y = 0$$

This equation pops up quite a bit in Physics. It is the equation for the motion of waves on the head of a drum when it it struck by a drumstick.
The Bessel function solution is

(26) $$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{1}{2}x\right)^{(2m+\alpha)}$$ and in our case

(26') $$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1)} \left(\frac{1}{2}x\right)^{(2m)}$$

The gamma function is defined as :

$$\Gamma(z) \equiv Lim_{n\to\infty} \frac{1 \bullet 2 \bullet 3 \bullet 4 \bullet \bullet n}{z(z+1)(z+2)(z+3) \bullet \bullet (z+n)} n^2$$

Where z is any number. If m is an integer

$\Gamma(n)=(n-1)!$ The ! symbol means factorial for example 3!=3*2*1 = 6. Also 0! = 1

For m=4 for example

$\frac{1}{\Gamma(m-1)}=\frac{1}{3!}=\frac{1}{6}$. Therefore the J's are all infinite series. But so are the sines and cosines.

The first fews gammas are

$\Gamma(1)=1$

$\Gamma(2)=1$

$\Gamma(3)=2$

$\Gamma(n)=(n-1)!$

Homework calculate and plot $z,\Gamma(z)$ where z= 0.1, 0.2, 50.

Gamma has many other definitions it can also be expressed as an integral

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt$$

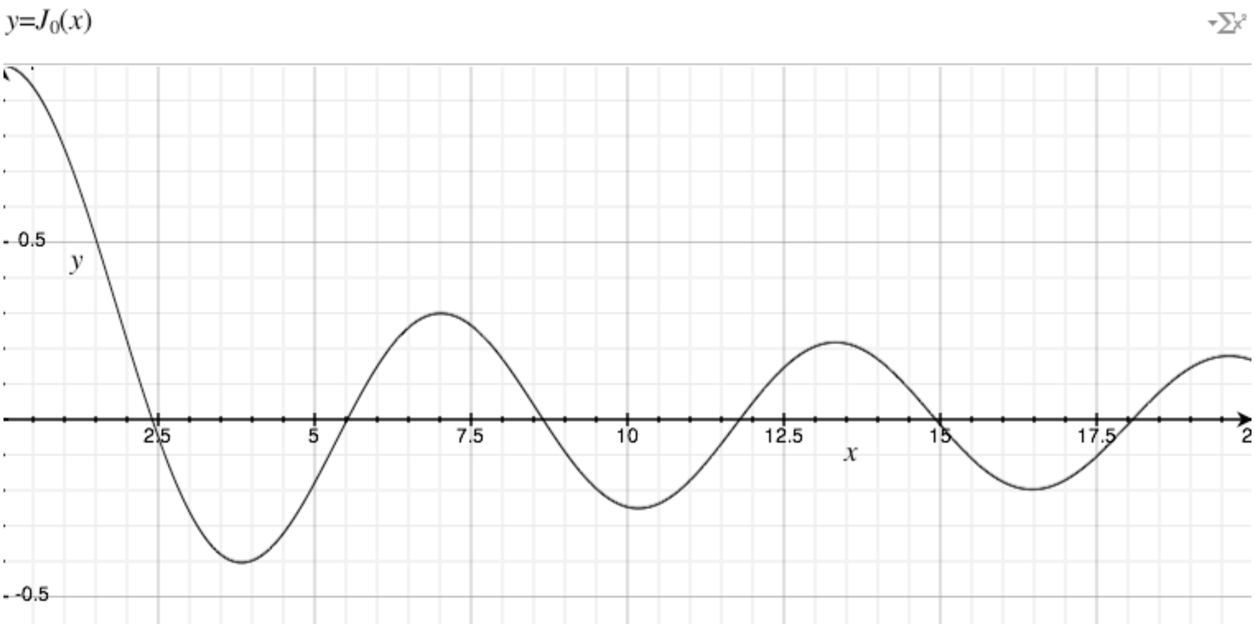
.

Homework plot the Jn's for various values of n. The places where J goes to zero are called the zeros of the Bessel functions. $J_0(k_\perp\rho)=0$. Find the first 20 and plot verses the number of the zero

The gamma functions have been studied for quite some time and one of their unusual properties is

$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin m\pi}$ where m is the same as in the above factorial definition.

Let's plot the zero order Bessel function



$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \quad ; \quad \sin x = \sum_{m=0}^{\infty} (-1)^m \frac{x^{(2m+1)}}{(2m+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} .... \quad ; \quad \cos x = \sum_{m=0}^{\infty} (-1)^m \frac{x^{(2m)}}{(2m)!}$$

IDL has all the Bessel functions built in!

Note another definition of the delta function is:

(27) $\delta(\rho - \rho') = \rho \int_0^{\infty} J_m(k\rho) J_m(k\rho') dk$

This is analogous to a similar integral for sines

$\int_0^{2\pi} \sin mx \sin nx \, dx = \pi \delta_{mn}$ Note the sines can used used in a Fourier series to build up any well behaved mathematical function. Since the Bessel functions behave in a similar fashion they too can be used in an infinite series to build up any function and in fact that is what we just did. They are part of the k space Fourier analysis of the potential. There is a special name for mathematical function that have the this property, they are called orthogonal functions. Aside from the sines and cosine there are an infinite amount of other orthogonal functions.