Magnetic field from a circular loop of wire

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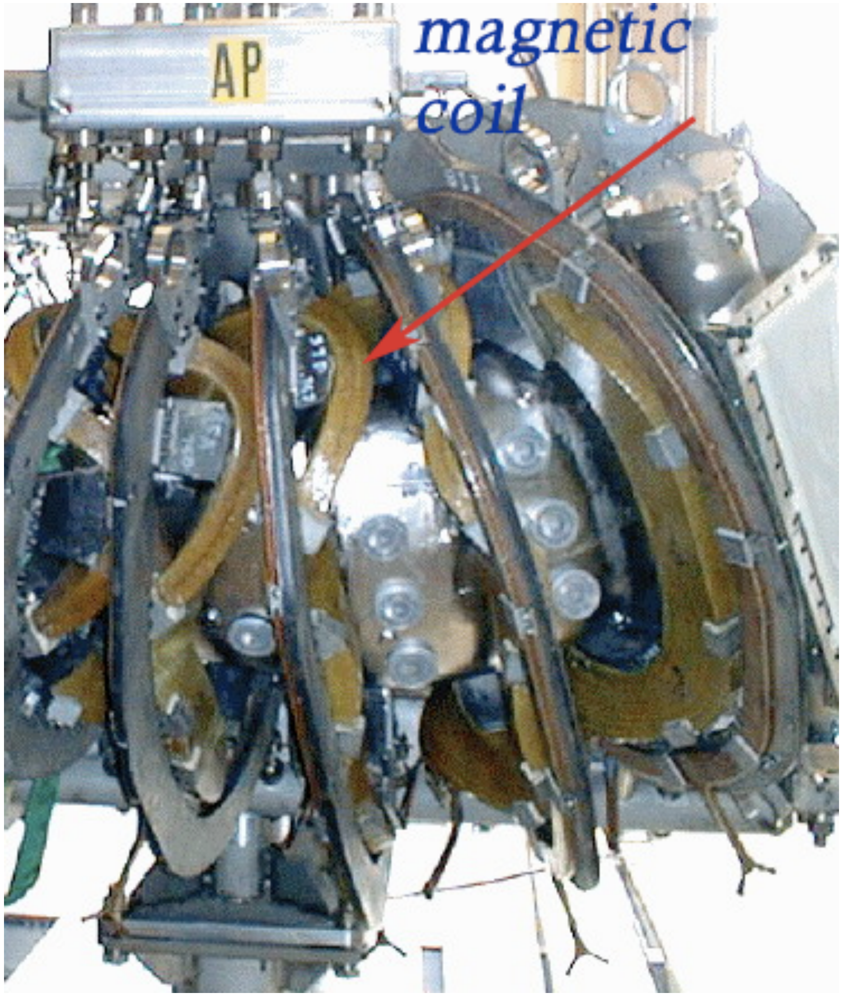
HSX-Stellerator

How can you calculate the magnetic field from coils as twisted as these?

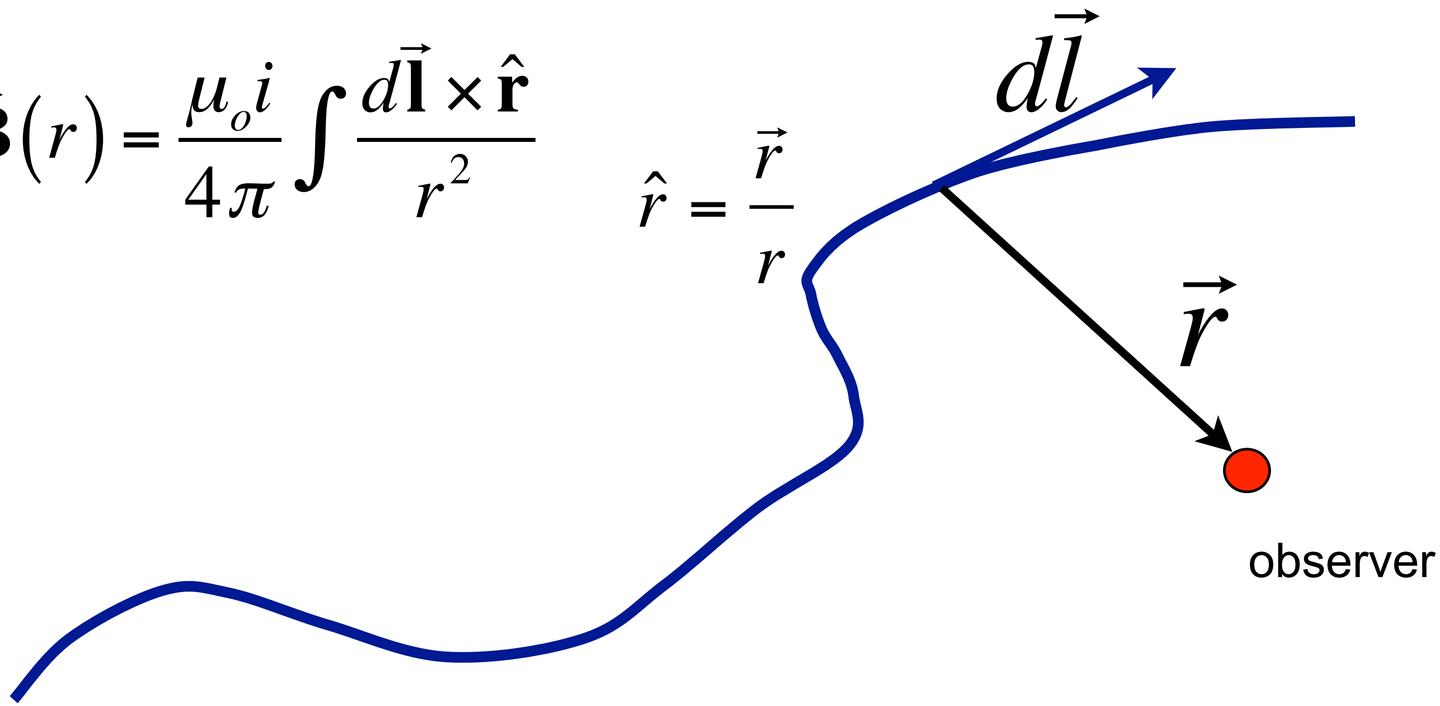


Law of Biot and Savart

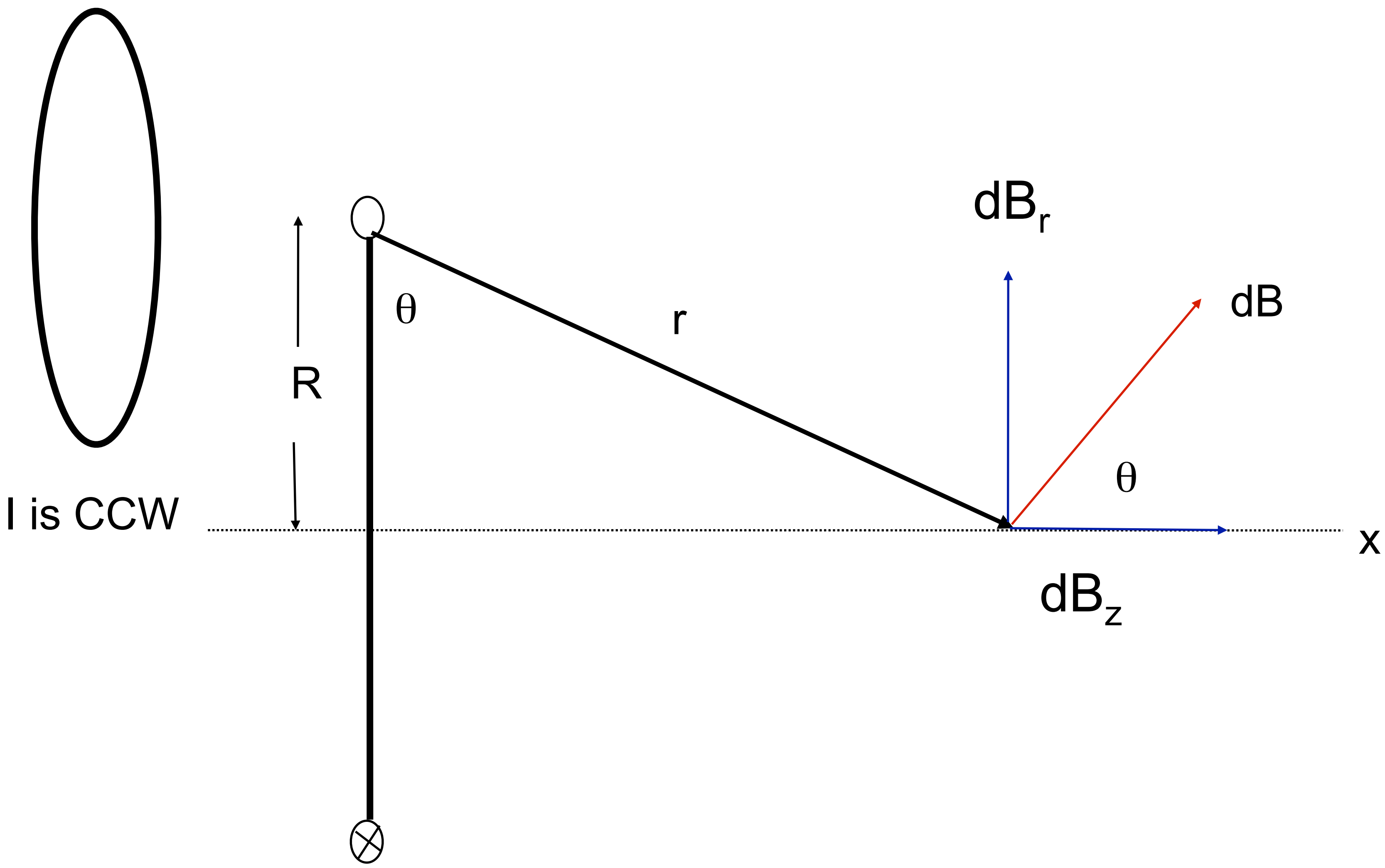
$$d\vec{\mathbf{B}}(r) = \frac{\mu_o i}{4\pi} \frac{d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}$$

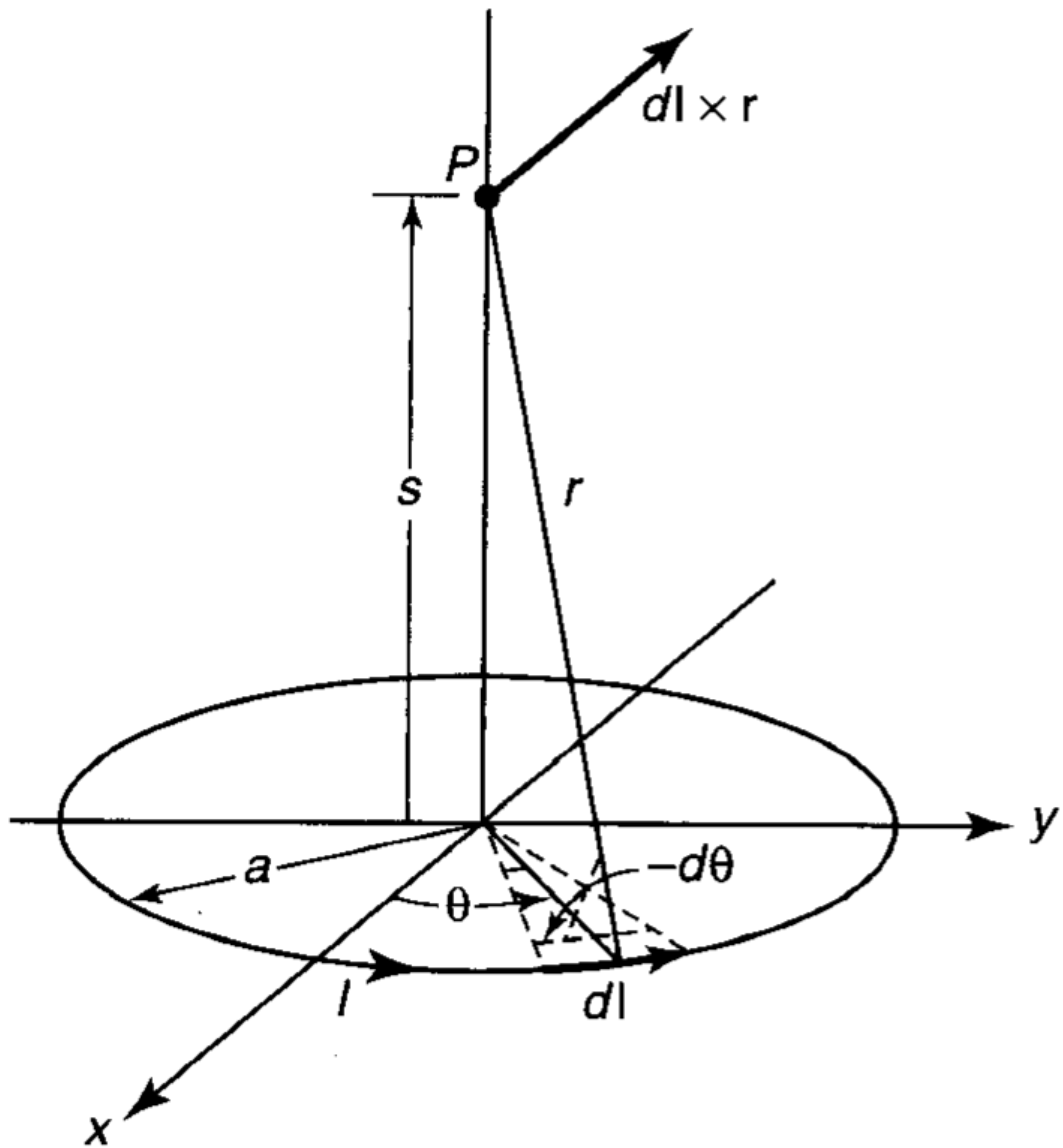
$$\vec{\mathbf{B}}(r) = \frac{\mu_o i}{4\pi} \int \frac{d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}$$

$$\hat{r} = \frac{\vec{r}}{r}$$



Current Loop on Axis B field





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$$d\vec{l}\times\vec{r}=ad\theta\left(-\sin\theta\hat{i}+\cos\theta\hat{j}\right)\times\left[\left(z-h\right)\hat{k}-a\cos\theta\hat{i}+a\sin\theta\hat{j}\right]$$

$$d\vec{l}\times\vec{r}=a\left(-\sin\theta\hat{i}+\cos\theta\hat{j}\right)\times\left[\left(z-h\right)\hat{k}-a\cos\theta\hat{i}+a\sin\theta\hat{j}\right]d\theta$$

$$d\vec{l}\times\vec{r}=a\left[\left(z-h\right)\sin\theta\hat{j}+\cos\theta\left(z-h\right)\hat{i}+a\sin^2\theta\hat{k}+a\cos^2\theta\hat{k}\right]$$

$$d\vec{l}\times\vec{r}=\left[a\left(z-h\right)\sin\theta\hat{j}+a\cos\theta\left(z-h\right)\hat{i}+a^2\hat{k}\right]$$

$$r=\sqrt{\left(z-h\right)^2+a^2}$$

$$\vec{\mathbf{B}}\left(r\right)=\frac{\mu_o i}{4\pi}\int_0^{2\pi}\frac{d\vec{\mathbf{l}}\times\hat{\mathbf{r}}}{r^2}$$

$$\vec{\mathbf{B}}(r) = \frac{\mu_o i}{4\pi}\int_0^{2\pi}\frac{d\vec{\mathbf{l}}\times\vec{r}}{r^3}$$

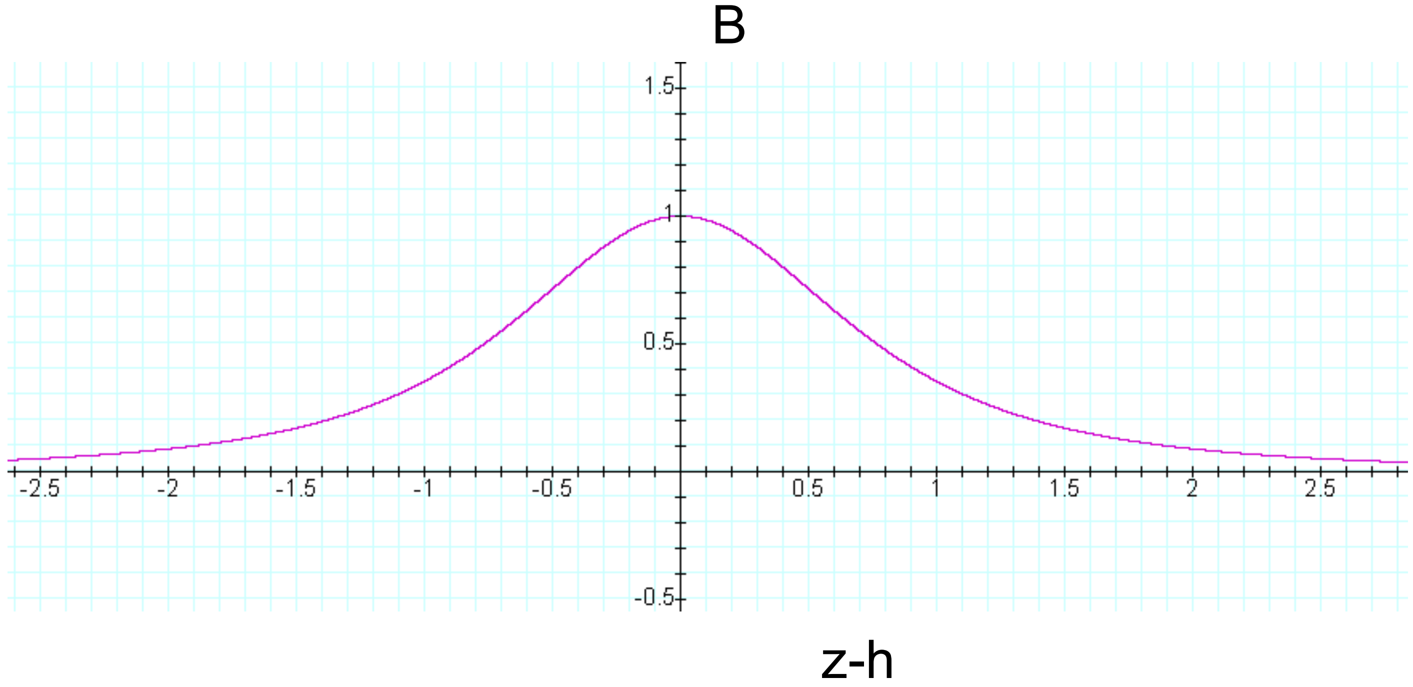
$$\vec{\mathbf{B}}(r) = \frac{\mu_o i}{4\pi}\int_0^{2\pi}\frac{\left[a(z-h)\sin\theta\,\hat{j} + a\cos\theta(z-h)\,\hat{i} + a^2\hat{k}\right]}{\left[(z-h)^2 + a^2\right]^{\frac{3}{2}}}$$

sine and cosine integrate to 0

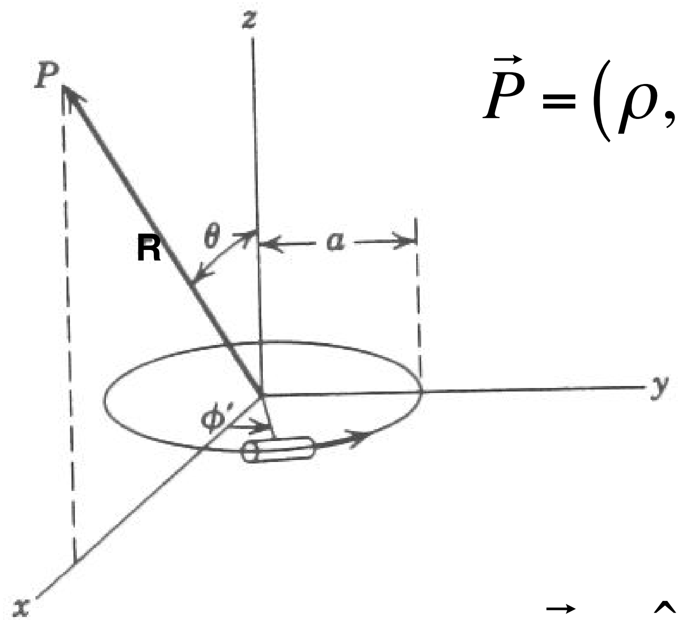
$$\vec{\mathbf{B}}(r) = \frac{\mu_o I}{4\pi}\int_0^{2\pi}\frac{\left[a^2\hat{k}\right]}{\left[(z-h)^2 + a^2\right]^{\frac{3}{2}}} = \frac{\mu_o I}{4\pi}2\pi\frac{a^2}{\left[(z-h)^2 + a^2\right]^{\frac{3}{2}}}$$

$$\vec{\mathbf{B}}(r) = \frac{\mu_o I}{2} \frac{a^2}{\left[(z-h)^2 + a^2\right]^{\frac{3}{2}}} \vec{\mathbf{k}}$$

B on axis (= B_x)



Current Loop off Axis B field

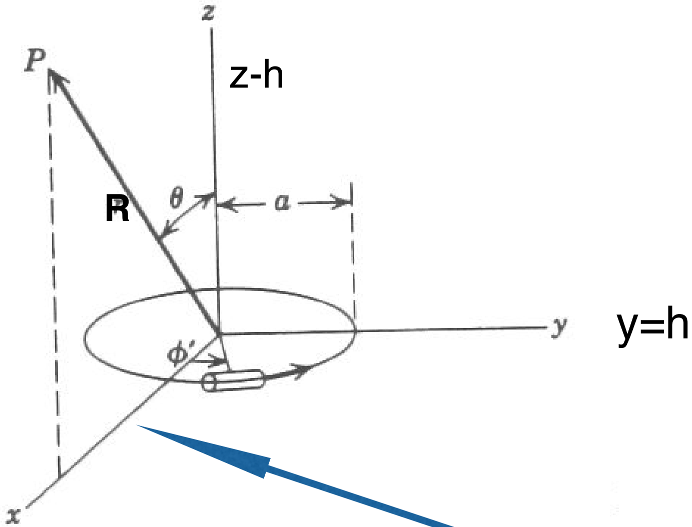


$$\vec{P} = (\rho, 0, h)$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{R}}{R^2}$$

Magnetic Vector Potential

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}}{R}$$



$$r - a\cos\varphi$$

$$R = \sqrt{\left(\left(r - a\cos\varphi\right)^2 + a^2\sin^2\varphi\right) + \left(z - h\right)^2}$$

$$R = \sqrt{a^2 + r^2 - 2ar\cos\phi + (z-h)^2}$$

$$d\vec{l} = -a\sin\phi d\phi\hat{i} + a\cos\phi d\phi\hat{j}$$

$$\vec{J} = -J_\varphi \sin\varphi\hat{i} + J_\varphi \cos\varphi\hat{j}$$

Since the problem is cylindrically symmetric about z we choose an observation point in the x-z plane, phi=0. Foe every phi the answer should be the same. There is one axis (z axis) of symmetry.

Putting dl and R into the vector potential :

$$\vec{A}=\frac{\mu_0 I}{4\pi}\left[\int_0^{2\pi}\frac{-a\sin\phi d\phi}{\sqrt{a^2+r^2-2ar\cos\phi+(z-h)^2}}\hat{i}+\frac{a\cos\phi d\phi}{\sqrt{a^2+r^2-2ar\cos\phi+(z-h)^2}}\hat{j}\right]$$

In the x-z plane (where P is) for every angle phi you get an equal but opposite contribution from -(phi) and the x component of A vanishes.

$$\vec{A}=\frac{\mu_0 I}{4\pi}\left[2\int_0^{\pi}\frac{a\cos\phi d\phi}{\sqrt{a^2+r^2-2ar\cos\phi+(z-h)^2}}\hat{j}\right]$$

$$\sqrt{a^2 + r^2 - 2ar\cos\phi + (z-h)^2} = \sqrt{(r+a)^2 + (z-h)^2 - 4ar\left(\frac{1+\cos\varphi}{2}\right)}$$

$$\phi \equiv \frac{\pi}{2} - 2\alpha$$

Define new angle alpha

$$\cos\varphi = \cos\frac{\pi}{2} - \cos(2\alpha) = -\cos(2\alpha) = -\left(1 - 2\sin^2\alpha\right)$$

$$\left(\frac{1+\cos\varphi}{2}\right) = \sin^2\alpha$$

$$\sqrt{(r+a)^2+(z-h)^2-4ar\left(\frac{1+\cos\varphi}{2}\right)}=\sqrt{(r+a)^2+(z-h)^2}\left[\sqrt{1-\frac{4ar\sin^2\alpha}{(r+a)^2+(z-h)^2}}\right]$$

$$k\equiv\sqrt{\frac{4ar}{(r+a)^2+(z-h)^2}}$$

define new variable, k
note k is dimensionless

$$k^2=\frac{4ar}{(r+a)^2+(z-h)^2};\quad \left((r+a)^2+(z-h)^2\right)k^2=4ar$$

$$\sqrt{\left((r+a)^2+(z-h)^2\right)}=2\frac{\sqrt{ar}}{k}$$

$$\sqrt{(r+a)^2+(z-h)^2-4ar\left(\frac{1+\cos\varphi}{2}\right)}=\frac{2\sqrt{ar}}{k}\sqrt{1-k^2\sin^2\alpha}$$

$$\phi \equiv \frac{\pi}{2} - 2\alpha, \qquad \mathrm{d}\phi\text{=-}2\mathrm{d}\alpha$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\left[2\int_0^{\frac{\pi}{2}} \frac{a\cos\phi d\phi}{\sqrt{a^2 + r^2 - 2ar\cos\phi + (z-h)^2}}\hat{\varphi}\right] =$$

$$\frac{\mu_0 Ia}{\pi}\int_0^{\frac{\pi}{2}} \frac{\left(1 - 2\sin^2\alpha\right)d\alpha}{2\left[\frac{\sqrt{ar}}{k}\right]\left(1 - k^2\sin^2\alpha\right)}\hat{\varphi} \;\; ; \;\; \mathrm{d}\phi\text{=-}2d\alpha$$

$$\vec{A} = \frac{\mu_0 Ik}{2\pi}\sqrt{\frac{a}{r}}\left[\int_0^{\frac{\pi}{2}} - \frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}}\vec{\phi} + \int_0^{\frac{\pi}{2}} \frac{2\sin^2\alpha d\alpha}{\sqrt{1 - k^2\sin^2\alpha}}\vec{\phi}\right]$$

These integrals cannot be obtained in closed form. They are elliptic integrals which are in fact a form of power series.

Elliptic Integrals

$$K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

Elliptic Integral of the first kind

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \alpha}\, d\alpha$$

Elliptic Integral of the second kind

Elliptic Integrals

The first integral in A was K but the second is not.
To find it we have to evaluate the following derivative.
Note the integral is over alpha not k!

$$\frac{\partial K(k)}{\partial k} = \frac{\partial}{\partial k}\int_0^{\frac{\pi}{2}} \frac{k\sin^2\alpha d\alpha}{\left(1-k^2\sin^2\alpha\right)} = \frac{E}{k\left(1-k^2\right)} - \frac{K}{k}$$

$$\frac{\partial E}{\partial k} = \frac{1}{k}\left(E-K\right)$$

$$\vec{A} = \frac{\mu_0 Ik}{2\pi}\sqrt{\frac{a}{r}}\left[\int_0^{\frac{\pi}{2}} -\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}\vec{\phi} + \int_0^{\frac{\pi}{2}} \frac{2\sin^2\alpha d\alpha}{\sqrt{1-k^2\sin^2\alpha}}\vec{\phi}\right]$$

$$\vec{A} = -\frac{\mu_0 Ik}{2\pi}\sqrt{\frac{a}{r}}\left[K + \frac{2}{k}\frac{\partial E}{\partial k}\right]\hat{\varphi}$$

$$\frac{\partial E}{\partial k} = \frac{1}{k}\left(E - K\right)$$

$$\vec{A} = -\frac{\mu_0 Ik}{2\pi}\sqrt{\frac{a}{r}}\left[K + \frac{2}{k^2}\left(E(k) - K(k)\right)\right]\hat{\varphi}$$

$$\vec{A} = \frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[-kK(k) - \frac{2}{k}\left(E(k) + K(k)\right)\right]\hat{\varphi}$$

$$\vec{A} = \frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\left(\frac{2}{k} - k\right)K(k) - \frac{2}{k}E(k)\right]\hat{\varphi}$$

$$\vec{B} = \nabla \times \vec{A}$$

In cylindrical coordinates

$$\nabla \times \vec{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\phi} + \frac{1}{r} \left(\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right) \hat{z}$$

A only has a phi component (because the current is in the phi direction.

$$\nabla \times \vec{A} = \left(-\frac{\partial A_\phi}{\partial z} \hat{r} - \frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} \hat{z} \right)$$

$$\frac{\partial A}{\partial z} = \frac{\partial}{\partial z}\left[\frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\left(\frac{2}{k}-k\right)K(k)-\frac{2}{k}E(k)\right]\right]$$

$$-\frac{\partial A}{\partial z} = -\left[\frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\left(-\frac{2}{k^2}-1\right)\frac{\partial k}{\partial z}K(k)+\left(\frac{2}{k}-k\right)\frac{\partial K(k)}{\partial z}-\frac{2}{k}\frac{\partial E(k)}{\partial k}+\frac{2}{k^2}E(k)\frac{\partial k}{\partial z}\right]\right]$$

so what is dk/dz?

$$k = \frac{\sqrt{4ar}}{\sqrt{\left((r+a)^2+(z-h)^2\right)}}$$

$$\frac{\partial k}{\partial z} = -\frac{2\sqrt{4ar}\,(z-h)}{\left((r+a)^2+(z-h)^2\right)^{\frac{3}{2}}} = -\frac{k^3(z-h)}{4ar}$$

also from chain rule for derivatives:

$$\frac{\partial K(k)}{\partial z} = \frac{\partial K(k)}{\partial k}\frac{\partial k}{\partial z} = -\frac{k^3(z-h)}{4ar}\frac{\partial K(k)}{\partial k}$$

$$samefor \quad \frac{\partial E(k)}{\partial z}$$

Now for some algebra:

$$-\frac{\partial A}{\partial z}=-\left[\frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\left(-\frac{2}{k^2}-1\right)\frac{\partial k}{\partial z}K(k)+\left(\frac{2}{k}-k\right)\frac{\partial K(k)}{\partial z}-\frac{2}{k}\frac{\partial E(k)}{\partial k}+\frac{2}{k^2}E(k)\frac{\partial k}{\partial z}\right]\right]$$

$$-\frac{\partial A}{\partial z}=-\left[\frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{r}}\left[\begin{array}{l}\left(-\frac{2}{k^2}-1\right)\left[-\frac{k^3(z-h)}{4ar}\right]K(k)+\left(\frac{2}{k}-k\right)\left[-\frac{k^3(z-h)}{4ar}\frac{\partial K(k)}{\partial k}\right]\\ -\frac{2}{k}\left[-\frac{k^3(z-h)}{4ar}\frac{\partial K(k)}{\partial k}\right]+\frac{2}{k^2}E(k)\left[-\frac{k^3(z-h)}{4ar}\right]\end{array}\right]\right]$$

$$-\frac{\partial A}{\partial z}=\left[\frac{\mu_0 I}{2\pi}\left[\frac{k^3(z-h)}{4ar}\right]\sqrt{\frac{a}{r}}\left[\begin{array}{l}\left(-\frac{2}{k^2}-1\right)K(k)+\left(\frac{2}{k}-k\right)\left[\frac{\partial K(k)}{\partial k}\right] \\ -\frac{2}{k}\left[\frac{\partial K(k)}{\partial k}\right]+\frac{2}{k^2}E(k)\end{array}\right]\right]$$

$$-\frac{\partial A}{\partial z}=\left[\frac{\mu_0 I}{2\pi}\left[\frac{k^3(z-h)}{4}\right]\sqrt{\frac{1}{ar^3}}\left[-k\left[\frac{\partial K(k)}{\partial k}\right]+\frac{2}{k^2}\left(E(k)-\left(1+\frac{2}{k^2}\right)K(k)\right)\right]\right]$$

$$\frac{\partial K(k)}{\partial k}=\frac{E}{k\left(1-k^2\right)}-\frac{K}{k}$$

$$-\frac{\partial A}{\partial z}=\left[\frac{\mu_0 I}{2\pi}\left[\frac{k^3(z-h)}{4}\right]\sqrt{\frac{1}{ar^3}}\left[-k\left(\frac{E}{k\left(1-k^2\right)}-\frac{K}{k}\right)+\frac{2}{k^2}\left(E(k)-\left(1+\frac{2}{k^2}\right)K(k)\right)\right]\right]$$

$$-\frac{\partial A}{\partial z}=\left[\frac{\mu_0 I}{2\pi}\left[\frac{k^3(z-h)}{4}\right]\sqrt{\frac{1}{ar^3}}\left[\left(-\frac{E(k)}{\left(1-k^2\right)}+K(k)\right)+\frac{2}{k^2}\left(E(k)-\left(1+\frac{2}{k^2}\right)K(k)\right)\right]\right]$$

$$=\left[\frac{\mu_0 I}{2\pi}\left[\frac{k^3(z-h)}{4}\right]\sqrt{\frac{1}{ar^3}}\left[E(k)\left[\frac{2}{k^2}-\frac{1}{\left(1-k^2\right)}\right]+K(k)\left(\frac{2}{k^2}\right)\right]\right]$$

Finally

$$-\frac{\partial A}{\partial z} = \left[\frac{\mu_0 I k\left(z-h\right)}{4\pi}\sqrt{\frac{1}{ar^3}}\left[K\left(k\right)-E\left(k\right)\left[\frac{2-k^2}{2\left(1-k^2\right)}\right]\right]\right]$$

what about? $-\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}$

$$rA_\phi = \frac{\mu_0 I}{2\pi}\sqrt{ra}\left[\left(\frac{2}{k}-k\right)K\left(k\right)-\frac{2}{k}E\left(k\right)\right]$$

$$\frac{\partial}{\partial r}\left(rA\right) = A + r\frac{\partial A}{\partial r}$$

After a similar amount of derivatives and algebra (try it)

$$\frac{\partial(rA_\phi)}{\partial r} = \frac{\mu_0 I}{2\pi}\frac{\partial}{\partial r}\left(\frac{1}{2}\sqrt{\frac{a}{r}}\right)\left[\left(\frac{2}{k}-k\right)K(k)-\frac{2}{k}E(k)\right]-\sqrt{ra}\,\frac{\partial}{\partial r}\left[\left(\frac{2}{k}-k\right)K(k)-\frac{2}{k}E(k)\right]$$

we need:

$$\frac{\partial k}{\partial r},\frac{\partial E(k)}{\partial r},\frac{\partial K(k)}{\partial r}$$

$$\frac{\partial k}{\partial r} = \frac{\partial}{\partial r}\sqrt{\frac{4ar}{\left(r+a\right)^2+\left(z-h\right)^2}} = \frac{\partial}{\partial r}\frac{(4ar)^{\frac{1}{2}}}{\left(\left(r+a\right)^2+\left(z-h\right)^2\right)^{\frac{1}{2}}}$$

after much algebra

$$\frac{\partial k}{\partial r} = \frac{k}{2r}\left(1-\frac{k^2\left(r+a\right)}{2a}\right)$$

$$\frac{\partial K\left(k\right)}{\partial r} = \frac{\partial K\left(k\right)}{\partial k}\frac{\partial k}{\partial r} = \frac{k}{2r}\left(1-\frac{k^2\left(r+a\right)}{2a}\right)\frac{\partial K\left(k\right)}{\partial k}$$

After more and more algebra

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rA\right)=\frac{\mu_0 Ikr}{4\pi\sqrt{ar^3}}\left[K+\frac{k^2\left(r+a\right)-2r}{2r\left(1-k^2\right)}E\right]$$

Finally

$$B(r,z) = \frac{\mu_0 I k}{4\pi\sqrt{ar^3}}\left\{\left[rK + \frac{k^2(r+a)-2r}{2(1-k^2)}E(k)\right]\hat{z} + \left[-(z-h)\left[K(k)-E(k)\left[\frac{2-k^2}{2(1-k^2)}\right]\right]\hat{r}\right]\right\}$$

$$k = \frac{\sqrt{4ar}}{\sqrt{\left((r+a)^2+(z-h)^2\right)}}$$

What happens on axis? r=0, k=0

$$B(r=0,z)=\frac{\mu_0 Ik}{4\pi\sqrt{ar^3}}\left\{\left[\frac{k^2(a)}{2(1-k^2)}E(0)\right]\hat{z}+\left[-(z-h)\left[K(0)-E(0)\left[\frac{2-k^2}{2(1-k^2)}\right]\right]\hat{r}\right]\right\}$$

$$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\alpha}\,d\alpha \quad ; \quad K(k)=\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

$$E(0)=\int_0^{\frac{\pi}{2}}d\alpha=\frac{\pi}{2}$$

$$K(0)=E(0)$$

$$B(r=0,z)=\frac{\mu_0 Ik}{8\sqrt{ar^3}}\left\{\left[\frac{k^2 a}{2(1-k^2)}\right]\hat{z}+\left[-(z-h)\left[1-\left[\frac{2-k^2}{2(1-k^2)}\right]\right]\hat{r}\right]\right\}$$

$k=0$ thus:

$$B(r=0,z)=\frac{\mu_0 I}{8\sqrt{ar^3}}\left[\frac{k^3 a}{2(1-k^2)}\right]\hat{z}$$

We better be careful k=0 and r=0!

$$B\left(r=0,z\right)=\frac{\mu_0 Ika}{8\sqrt{a}\sqrt{r^3}}\left[\frac{4ar\sqrt{ar}}{\left(\left(r+a\right)^2+\left(z-h\right)^2\right)^{\frac{3}{2}}}\frac{1}{\left(1-k^2\right)}\right]\widehat{z} \quad ;;\ k^2=\frac{4ar}{\left(\left(r+a\right)^2+\left(z-h\right)^2\right)}\vec{r}$$

$$B\left(r=0,z\right)=\frac{\mu_0 Ik}{2\sqrt{r^3}}\left[\frac{a^2\sqrt{r^3}}{\left(\left(r+a\right)^2+\left(z-h\right)^2\right)^{\frac{3}{2}}}\frac{1}{\left(1-k^2\right)}\right]\widehat{z}$$

Now with k=0 and r=0

$$B\left(r=0,z\right)=\frac{\mu_0 Ia^2}{2}\left[\frac{1}{\left(\left(a\right)^2+\left(z-h\right)^2\right)^{\frac{3}{2}}}\right]\widehat{z}$$

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;
********************ELLIPTIC************************************
; Calculates elliptic integrals using formulae from Abramovitz and Stegun
pro elliptic,kk2, E, K
; note kk2 is 4*a*R/s^2 where a is the radius of the coil, R is the
; radial point of observation, z is the axial point of observation ;
;(relative to the origin)
; s = sqrt[(a+r)^2+z^2]
;
     eta = 1 - kk2& eta2 = eta ^ 2& ln1eta = -aLOG(eta)
     a0 = 1.3862944& b0 = .5
     a1 = .1119723& b1 = .1213478
     a2 = .0725296& b2 = .0288729

     K = a0 + a1 * eta + a2 * eta2 + (b0 + b1 * eta + b2 * eta2) * ln1eta

     a1 = .4630151& b1 = .2452727
     a2 = .1077812& b2 = .0412496

     E = (1 + a1 * eta + a2 * eta2) + (b1 * eta + b2 * eta2) * ln1eta
return
END
```