1/5/16 Algebraic Number Theory 2A

$e^{\pi\sqrt{163}} = ,99999$

$$\begin{array}{cccc} \mathbb{Q} & \mathbb{Q}(\sqrt{-5}) & \mathbb{Q}\sqrt{-m} & \mathbb{C} \\ | & | & m>0 & | \\ \mathbb{Z} & \mathbb{Z}[\sqrt{-5}] & & \mathbb{Q} \end{array}$$

$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$

① Algebraic Number $\sqrt{2}$, $\sqrt[4]{17}$, $\sqrt{2} + \sqrt{11}$, $\cos 2\ldots$

② Algebraic Integer sub domain $+ - \times$

$x^7 + 3x + 14 = 0$

③ Algebraic #'s form a field

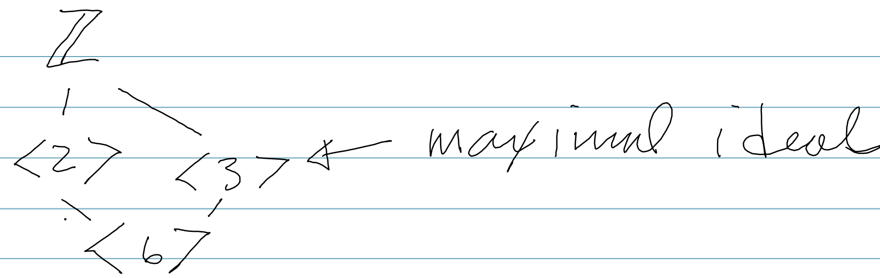
$+, -, \cdot$

④ Sub domain
irreducible

Prime element $p \mid ab \Rightarrow p \mid a$ or $p \mid b$

every prime is irreducible.

$$\begin{array}{c} D \\ | \\ I \\ \{0\} \end{array}$$ Proper Ideal



Prime ideal $ab \in I \Rightarrow a \in I$ or $b \in I$

$JK \subseteq I \Rightarrow J \subseteq I$ or $K \subseteq I$

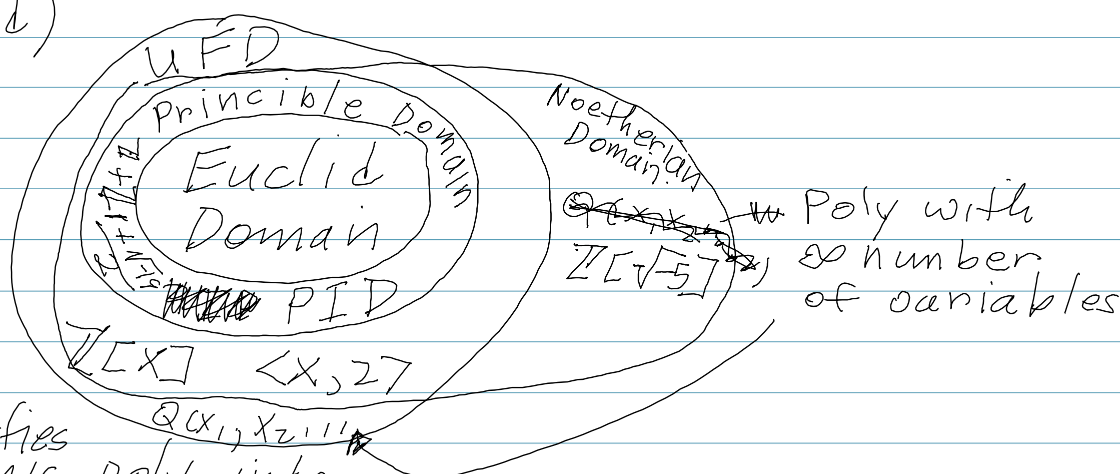
I is prime ideal of D iff D/I

$\langle 6\rangle$	$1+\langle 6\rangle$	$2+\langle 6\rangle$	$3+\langle 6\rangle$	$4+\langle 6\rangle$	$5+\langle 6\rangle$

cosets

I is maximal ideal D/I is a field.

(side board)



Alg Int satisfies monic poly integ coeff,

$x^3 + 3x^2 - \sqrt{3}x + i$

Thrm: Let $\alpha \in \mathbb{C}$ satisfy a monic polynomial whose coefficients are algebraic integer. Then α is an algebraic integer.

if $\alpha = 3$

powers $1, 3, 3^2, 3^3, 3^4 \dots$

generate additive subgroup.

if $\alpha = \frac{1}{2}$

$2x - 1 = 0$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ do not generate finite additive group.

Prf: Say $\alpha^n + C_{n-1}\alpha^{n-1} + \dots + C_1\alpha + C_0 = 0$

where C_i are algebraic integers.

The c_i generate a subdomain D ring $+\times$

See that the powers of α generate

~45 min a D-submodule M spanned by $1, \alpha, \alpha^2, \dots, \alpha$

Know that the powers of C_i lie inside a finitely-generated additive group

G_i, with generators γ_{ij} $(1 \le j \le n_i)$
Follow that M lies inside the additive group generated by all elements
$\gamma_{1j_1}, \gamma_{2j_2} \cdots \gamma_{n-1\, j_{n-1}} \cdots \alpha^k$ where
$1 \le j_i \le n_i$, $0 \le i \le n-1$, $0 \le k \le n-1$.
This set of elements is finite.
Thus M is contained as a subgroup of a finitely-generated abelian group, so is itself finitely-generated
Conclude α is an algebraic integer

$\mathbb{Q}(\sqrt{2}, \sqrt{3})$ Consider: $\quad$ $\text{Norm}(a+b\sqrt{m}) = a^2 - mb^2$

$\{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}\}$ ⓐ $\mathbb{Q}(\sqrt{15})$: $2 \cdot 5 = (5+\sqrt{15})(5-\sqrt{15})$

$$2 = (a+b\sqrt{15})(c+d\sqrt{15})$$

$= \mathbb{Q}(\sqrt{2}+\sqrt{3})$

$$4 = (a^2 - 15b^2)(c^2 - 15d^2)$$

$\sim$(10 min)

$$a^2 - 15b^2 = 2 \; ?$$

mod 3

$$a^2 \equiv 2 \pmod 3$$

$a^2 - 15b^2 = -2$

mod 5

$a^2 \equiv 3 \pmod 5$

(a) $\mathbb{Q}\sqrt{15} = 2 \cdot 5 = (5+\sqrt{15}$ $\qquad \sqrt{5}$

(b) $\mathbb{Q}\sqrt{30}$: $2 \cdot 3 = (6+\sqrt{30})(6-\sqrt{30})$ $\qquad \sqrt{3}$

(c) $\mathbb{Q}\sqrt{-10}$ $\quad 2 \cdot 7 = (2+\sqrt{-10})(2-\sqrt{-10})$ $\qquad \sqrt{2}$

Note: $5+\sqrt{15} = \sqrt{5}(\sqrt{5}+\sqrt{3})$

$5-\sqrt{15} = \sqrt{5}(\sqrt{5}-\sqrt{3})$

multiply and cancel "5"

(a) $2 = (\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$

See two factorizations given above for 10 are obtained by grouping the factors in $\sqrt{5}\sqrt{5}(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$

~70 min in ~~[illegible]~~ 2 ways.

See that inside $\mathbb{Q}(\sqrt{3},\sqrt{5}) = \{a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15} \mid$

$a,b,c,d \in \mathbb{Q}$

$I+J=\{i+j\}$ Product of 2 ideals $IJ=\{i_1j_1+\dots+i_nj_n \mid i_k\in I, j_\ell\in J\}$

Let D = integral domain.

Say $x, a, b, \in D$ Say $x=ab$

See $\langle x\rangle=\langle a\rangle\langle b\rangle$

$6=2\cdot 3$
$=3\cdot 2$
$=-2\cdot -3$
$=-3\cdot -2$

Generalize $x=p_1p_2\cdots p_n$

See $\langle x\rangle=\langle p_1\rangle\langle p_2\rangle\cdots\langle p_n\rangle$

$2=(1+i)(1-i)$
$=(-1+i)(-1-i)$

$\langle a\rangle=\langle au\rangle$

~80 min $ra=ru^{-1}(au)$

$\mathbb{Q}(\sqrt{m})$

$\mathbb{Q}(\sqrt{2})$

Consider again:

$(1+\sqrt{2})(-1+\sqrt{2})=1$

In ~~$\mathbb{Q}(\sqrt{3}\ldots$~~ $\mathbb{Q}(\sqrt{5},\sqrt{3})$ have

$(1+\sqrt{2})^{10}=a+b\sqrt{2}$

$\langle 10\rangle=\langle\sqrt{5}\rangle\langle\sqrt{5}\rangle\langle\sqrt{5}+\sqrt{3}\rangle\langle\sqrt{5}-\sqrt{3}\rangle$

$\mathbb{Q}(\sqrt{3},\sqrt{5})$. integers in $\mathbb{Q}(\sqrt{3},\sqrt{5})$

$|\ 2$

$\mathbb{Q}(\sqrt{15})$ $\mathbb{Z}[\sqrt{15}]=\mathbb{Z}+\mathbb{Z}(\sqrt{15})$

if $IJ=K$

$(I\cap L)(J\cap L)=K\cap L$

$|\ 2$

? $1=(\sqrt{3}+\sqrt{5})(a+b\sqrt{3}+c\sqrt{5}+d\sqrt{15})$

$\mathbb{Q}$ $\mathbb{Z}$

Note: Let $I=\langle\sqrt{5}+\sqrt{3}\rangle\cap\mathbb{Z}[\sqrt{15}]$

$\frac{1}{\sqrt{3}+\sqrt{5}}$ can not be ~~done~~.

Note $\sqrt{3}(\sqrt{5}+\sqrt{3}) = 3+\sqrt{15} \in I$

$\sqrt{5}(\sqrt{5}+\sqrt{3}) = 5+\sqrt{15} \in I$

subtract $2 \in I$

Break end Lect 1A Algebraic Number Theory 2.

<u>After Break</u>

1/5/16 Lect 1B

$I = \langle \sqrt{5}+\sqrt{3} \rangle \cap \mathbb{Z}[\sqrt{15}]$ saw $2 \in I$

Say $I = \langle a+b\sqrt{15} \rangle = \langle k \rangle$ where $a,b \in \mathbb{Z}$

Says $a^2-15b^2 \mid 4$ k is unit

If norm$(a+b\sqrt{15}) = \pm 1$, Then k is un T

which says $\langle k \rangle = \mathbb{Z}[\sqrt{15}]$

9pm Forces $1 \in I$ But $1 \notin \langle \sqrt{3}+\sqrt{5} \rangle$ contradiction

by argument to show that $\frac{1}{\sqrt{3}+\sqrt{5}}$ is <u>not</u> an algebraic integer. And so on...

<u>Note</u> $N(5+\sqrt{15}) = 10$

$N(3+\sqrt{15}) = -6$

Since each # is multiple of k

see $N(k) \mid 10$ $N(k) \mid -6$