Kinetic Theory of the Ponderomotive Effects in a Plasma

Vladimir B. Krapchev
Plasma Fusion Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 25 September 1978)

Large-amplitude, spatially modulated coherent high-frequency waves produce a change in the plasma equilibrium. The nonresonant wave-particle interaction is investigated and the local, time-independent Vlasov distribution function is found for the one-dimensional case to all orders in the electric field amplitude. For a critical value of the electric field amplitude at a fixed temperature, the distribution function exhibits "double hump" but the Penrose criterion shows that it is stable.

In laser-pellet experiments and rf heating in tokamaks, large-amplitude high-frequency waves are launched into the plasma. These waves will be spatially modulated by the finite extent of the external source, even for homogeneous plasmas. If the phase velocity of the waves is much larger than the thermal velocity the resonant effects (Landau damping trapping) are negligible. The modulation of the amplitude is crucial in changing the plasma equilibrium. Only recently[1,2] it was pointed out that the phase-space dependence of the distribution function is modified in a nontrivial manner. However, there has been no success in finding an exact solution of the Vlasov equation even in the one-dimensional case. In this paper I present a solution which is valid for a general form of the modulation and arbitrarily large amplitudes of the electric field.

The ponderomotive effect is present when $L/T \ll v$, where L is the scale length of the modulation, T is the time scale (time Tayloring, pump depletion) and v is the velocity of a particle. For typical laser and rf experiments $L/T \simeq 0.01 v_{Te}$ and my analysis is valid for nearly all of the phase space. In the homogeneous limit $L/T \gg v$ and no ponderomotive density change occurs. We shall find the solution in both limits, but not for an arbitrary value of vT/L. Therefore, a smooth transition between the ponderomotive distribution function and the homogeneous limit cannot be obtained.

First we look for a local, time-independent distribution function $f_0 \equiv f_0(v, v_0(x))$, where $v_0(x) = qE_0(x)/m\omega_0$ and ω_0 is the frequency of the pump. f_0 describes the steady state on a time scale much shorter than the collisional time scale. In the limit $v_0 \to 0$, f_0 becomes the Maxwellian distribution. The general solution of the Vlasov equation will depend also on all derivatives of $v_0(x)$. However, these contributions are smaller by a factor of $v/L\omega_0 \ll 1$. Our solution is exact in the sense of being valid for v_0/v_T arbitrarily large. A general form of the modulation function $v_0(x)$ includes all the realistic cases where $v_0(x) \neq 0$ for a finite region of space and $v_0(x_B) = 0$ on the boundary of that region. x_B could extend to $\pm\infty$ if $v_0^2(x)$ is integrable in all space. This allows us to decouple the Vlasov-Maxwell system of equations and solve a truly nonlinear problem.

One can examine the ponderomotive change of the electron distribution function and neglect the effect on the ions, which is smaller by the mass ratio m/M. This is not completely consistent and one should introduce an ambipolar potential due to charge separation. Only when the ponderomotive and ambipolar potentials are balanced to produce charge neutrality can the problem be easily resolved. I assume that this is the case.

For simplicity one uses the dipole approximation for the pump. This assumption is justified when $\omega_0/k_0 \gg v_T$. The finite k_0 of the high-frequency wave does not lead to an important change in the plasma state. The corrections are of the order $k_0 v_T/\omega_0$ for the one-dimensional example. However, in a realistic three-dimensional case the effect of a finite k_0 leads to the generation of a steady-state current and a magnetic field.

The Vlasov equation for electrons in an externally driven high-frequency field ($k_0 = 0$) is

$$\partial f/\partial t = v\nabla f = \omega_0 v_0(x)\cos(\omega_0 t)\nabla_u f. \tag{1}$$

If the field is not modulated [$v_0(x) = \text{const}$], the solution of (1) is given by a displaced Maxwellian distribution

$$f = \frac{n_0}{v_T \pi^{1/2}} \exp\{-v_T^{-2}[v + v_0\sin(\omega_0 t)]^2\}. \tag{2}$$

The time-independent component of (2) is simply

$$f_0 \equiv \langle f\rangle = \frac{n_0}{2\pi v_T}\int_{-\infty}^{\infty} e^{ip\omega} e^{-p^2/4} J_0(p\omega_0)\,dp, \tag{3}$$

where the bracket $\langle\cdots\rangle$ denotes time average over a period $2\pi/\omega_0$, with $\omega = v/v_T$ and $\omega_0 = v_0/v_T$. This is the equilibrium in the oscillating frame analysis. Obviously, the kinematics will not change the properties of the plasma. For example, $n=n_0$, i.e., the density is undisturbed.

The nontrivial modification in the distribution function is due to the modulation of the field amplitude. The solution of Eq. (1) can be written in a Fourier series:

$$f=f_0+\sum_{n=1}^{\infty}[f_n\exp(-in\omega_0 t)+\text{c.c.}]. \tag{4}$$

By substituting (4) in (1) one obtains the following infinite set of equations:

$$(v/\omega_0)\nabla f_0=\tfrac{1}{2}v_0\nabla_v(f_1+f_1^*), \tag{5}$$

$$[-in+(v/\omega_0)\nabla]f_n=\tfrac{1}{2}v_0\nabla_v(f_{n-1}+f_{n+1}),\quad n\geq 1. \tag{6}$$

Since we consider $(v/\omega_0)\nabla f_n \ll f_n$, Eq. (6) can be simplified:

$$f_n=\frac{1}{-in}\left(1-\frac{i}{n}\frac{v\nabla}{\omega_0}\right)\tfrac{1}{2}v_0\nabla_v(f_{n-1}+f_{n+1}). \tag{7}$$

We shall find the Taylor series of f_0 in the neighborhood of $v_0=0$. The analytical continuation for all v_0 is performed by summing the series. This method of calculating f_0 is equivalent, in some sense, to the integration over exact particle trajectories. It has the advantage of being applicable for any form of the amplitude $v_0(x)$.

We shall calculate explicitly the series to v_0^4 and then suggest a general formula. After a rather lengthy algebra the result was checked to v_0^8. One should point out that it is not possible to derive a single equation for any of the Fourier components from the infinite set of differential equations. Thus, some mathematical intuition is always called for in breaking the chain of equations. To find f_0 to order v_0^2 substitute f_1 from (7) in (5) and neglect the contribution from f_2 (order v_0^4). The result is

$$w\nabla f_0^{(2)}=2(\tfrac{1}{2}w_0\nabla_w)w\nabla(\tfrac{1}{2}w_0\nabla_w)f_M, \tag{8}$$

where I have introduced the dimensionless variables w and w_0. One may note the appearance of a ponderomotive force term $\nabla(w_0^2)$. The concept of a force is not enough to describe the profound change in the phase-space dependence of the distribution function and is not very useful in the present analysis. If one considers (8) as a quasilinear equation for f_0 ($f_M, f_0^{(2)}\to f_0$) a solution is known.[1] The result is completely different from the exact solution, where the harmonics of all orders are taken into account. Formula (8) is a recurrence relation between $f_0^{(0)}=f_M$ on the right-hand side and $f_0^{(2)}$ to second order in w_0. An integration $\int_{x_B}^{x}dx$ of (8) and use of the boundary condition $w_0(x_b)=f_0^{(2)}(x_B)=0$ lead to

$$f_0^{(2)}=(-\tfrac{1}{2}w_0^2)(1-\tfrac{1}{2}\nabla_w^2)f_M. \tag{9}$$

The fourth-order term $f_0^{(4)}$ involves the second harmonic f_2. The lowest-order term for f_2 is [see (7) for $n=2$],

$$f_2=\frac{1}{-2i}\left(1-\frac{i}{2}\frac{v\nabla}{\omega_0}\right)\tfrac{1}{2}v_0\nabla_u f_1. \tag{10}$$

The third-harmonic term f_3 leads to contributions for f_0 in v_0^6. f_1 is determined to first order in v_0:

$$f_1=\frac{1}{-i}\left(1-i\frac{v\nabla}{\omega_0}\right)\tfrac{1}{2}v_0\nabla_u f_M. \tag{11}$$

One substitutes (11) in (10) to find f_2 in terms of f_M. Only the first-order terms in the operator $(v/\omega_0)\nabla$ are taken into account. This is the essence of the approximation, which leads to a local solution for f_0. f_0 will depend only on the field amplitude and all derivative terms are neglected. For f_2, one obtains

$$f_2=-\tfrac{1}{2}(\tfrac{1}{2}v_0\nabla_v)^2f_M-\frac{1}{4i\omega_0}\nabla(\tfrac{1}{2}v_0)^2(v\nabla_v^2+\nabla_v v\nabla_v)f_M. \tag{12}$$

From Eq. (7) for $n=1$ has

$$f_1+f_1^*=-\frac{1}{i}\frac{v_0}{2}\nabla_v(f_2-f_2^*)+\frac{v\nabla}{\omega_0}\left(\frac{v_0}{2}\right)\nabla_v(2f_0+f_2+f_2^*). \tag{13}$$

In (13) we substitute the result for f_2 from (12). From Eq. (5) one obtains, after some algebra, a recurrence relation for $f_0^{(4)}$:

$$w\nabla f_0^{(4)}=2(\tfrac{1}{2}w_0)^2\nabla_w w\nabla_w\nabla f_0^{(2)}+\nabla(\tfrac{1}{2}w_0)^2[\nabla_w w\nabla_w f_0^{(2)}-\tfrac{1}{2}(\tfrac{1}{2}w_0\nabla_w)^2(w\nabla_w^2+\nabla_w w\nabla_w)f_M - \tfrac{3}{2}(\tfrac{1}{2}w_0)^2\nabla_w w\nabla_w^3 f_M]. \tag{14}$$

Substitute in (14) the result for $f_0^{(2)}$ from (9) and after a straightforward calculation one can write

$$f_0^{(4)}=\tfrac{1}{2}(-\tfrac{1}{2}w_0^2)^2[\tfrac{1}{2}(\tfrac{1}{2}\nabla_w^2)^2-2(\tfrac{1}{2}\nabla_w^2)+1]f_M. \tag{15}$$

So far we have moved two steps up the infinite set of equations. However, some general features become apparent. One can write, to all orders in w_0,

$$f_0^{(2n)} = (1/n!)(-\tfrac{1}{2}w_0^2)^n P_n(\tfrac{1}{2}\nabla_w^2) f_M, \tag{16}$$

where P_n is polynomial function of the differential operator ∇_w^2. To identify this polynomial I have written the expressions in (9) and (15) in a suggestive form. The reader may verify that[3]

$$P_n(\tfrac{1}{2}\nabla_w^2) \equiv L_n(\tfrac{1}{2}\nabla_w^2), \tag{17}$$

where L_n is the Laguerre polynomial of order n. I have confirmed this result to order w_0^8 after an exceedingly long calculation. The Taylor series for f_0 can be written as

$$f_0(w, w_0) = \sum_{n=0}^{\infty} (1/n!)(-\tfrac{1}{2}w_0^2)^n L_n(\tfrac{1}{2}\nabla_w^2) f_M. \tag{18}$$

To sum the series I take the Fourier transform of f_0 in w:

$$\tilde{f}_0(p) = \int_{-\infty}^{\infty} e^{-ipw} \sum_{n=0}^{\infty} (-\tfrac{1}{2}w_0^2)^n (1/n!) L_n(-\tfrac{1}{2}p^2) f_M \, dw. \tag{19}$$

The series in the integrand can be easily summed[3] and leads to

$$\sum_{n=0}^{\infty} (1/n!)(-\tfrac{1}{2}w_0^2)^n L_n(-\tfrac{1}{2}p^2) = \exp(-\tfrac{1}{2}w_0^2) J_0(pw_0). \tag{20}$$

After a trivial integration in (19), the Fourier transform $\tilde{f}_0(p)$ becomes

$$\tilde{f}_0(p) = (n_0/v_T)\exp(-\tfrac{1}{2}w_0^2 - \tfrac{1}{4}p^2) J_0(pw_0). \tag{21}$$

Finally, the local time-independent "exact" solution is

$$f_0(w, w_0) = \exp(-\tfrac{1}{2}w_0^2)(n_0/2\pi v_T)\int_{-\infty}^{\infty} \exp(ipw)\exp(-p^2/4) J_0(pw_0)\, dp. \tag{22}$$

The plasma equilibrium has been found to all orders in the electric field amplitude. The formula is remarkably simple and, in fact, looks almost like the oscillating-frame result, except for the exponential dependence on w_0. However, it is this dependence that significantly changes the "ground" state of the plasma. I shall call f_0 in (22) the renormalized distribution function.

The density is given by

$$n = n_0 \exp(-\tfrac{1}{2}w_0^2). \tag{23}$$

This formula is well known from the investigation of rf confinement of plasmas.[4] The exponential profile modification is indeed consistent with the Vlasov equation. Furthermore, the temperature changes, because of the nonresonant diffusion:

$$\left[\int_{-\infty}^{\infty} mv^2 f_0(w, w_0)\, dw\right]\left[\int_{-\infty}^{\infty} f_0(w, w_0)\, dw\right]^{-1} = T_0(1 + w_0^2). \tag{24}$$

The most striking feature of the renormalized distribution function is the occurrence of a double hump above a certain critical value w_{0c}. In Fig. 1 I have plotted f_0 for $w_0 = 1$, 1.5, 2. One can see the appearance of a positive slope in the bulk of the distribution function. It seems that at a certain critical point the plasma is split into two beams. At $w_{0c} \simeq 1.25$ a second maximum develops. However, one can show that the double hump does not lead to an unstable equilibrium. With the minimum of the distribution function at $w = 0$ the Penrose function is

$$P = \int_{-\infty}^{\infty} [f_0(0, w_0) - f_0(w, w_0)] w^{-2}\, dw. \tag{25}$$

A simple integration with f_0 given by (22) shows

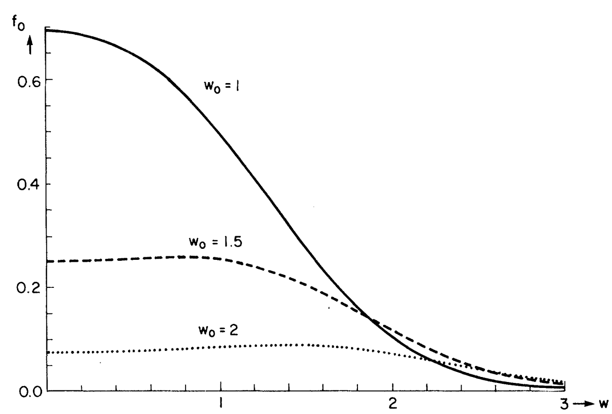

FIG. 1. Plot of $f_0(w, w_0)$ from Eq. (22) for $w_0 = 1$, 1.5, and 2. (Here $w = v/v_T$ and $w_0 = v_0/v_T$.)

that P is positive for all values of the normalized electric field amplitude w_0. However, instabilities may arise due to resonant mode coupling between the pump wave and waves from the thermal noise and the latter can grow to large amplitudes.

The homogeneous result is recovered in the limit $L/T \gg v$. In this case the infinite chain of relations from the Vlasov equation becomes

$$\frac{1}{\omega_0}\frac{\partial}{\partial t} f_0{}^H = \tfrac{1}{2}v_0(t)\nabla_v(f_{n-1}{}^H + f_{n+1}{}^{H*}), \tag{26}$$

$$f_n{}^H = \frac{1}{-in}\left(1 - \frac{i}{n}\frac{1}{\omega_0}\frac{\partial}{\partial t}\right)\tfrac{1}{2}v_0(t)\nabla_v(f_{n-1}{}^H + f_{n+1}{}^H), \quad n \geq 1. \tag{27}$$

A straightforward calculation gives the following infinite series for $f_0{}^H$:

$$f_0{}^H = \sum_{n=0}^{\infty} (n!)^{-2}(\tfrac{1}{2}w_0{}^2)^n(\tfrac{1}{2}\nabla_w{}^2)^n f_M = I_0(w_0\nabla_w)f_M, \tag{28}$$

where I_0 is the Bessel function of an imaginary argument. By summing the series in the same way as before, see Eqs. (18)–(22), one finds

$$f_0{}^H = \frac{n_0}{2\pi v_T}\int_{-\infty}^{\infty} e^{ipw} e^{-p^2/4} J_0(pw_0)\,dp. \tag{29}$$

This is the result of the oscillating-frame analysis, i.e., the homogeneous limit.

My derivation unifies two seemingly different approaches: the oscillating-frame analysis and the concept of a ponderomotive force. It is only through the Vlasov equation for modulated waves that their relationship can be properly established. The strong nonresonant wave-particle interactions ($w_0 > 1$) lead to a radical departure from the traditional results of linear and quasilinear theories.

To understand the nonlinear laser-plasma interactions the model above should be extended to include a modulated electromagnetic wave. In this case the field amplitude is elliptically polarized and is modulated in the plane of the ellipse. A finite k_0 of the pump should be included to describe the steady-state current generated along the direction of propagation. As an example of applications to heating in tokamaks examine the case of a lower-hybrid wave launched by a waveguide array. The nonlinear coupling can be described by the one-dimensional model. The external structure generates a field along the magnetic field lines and is modulated in the same direction. The velocity of oscillation across the magnetic field lines is small and the density modulation is well represented by Eq. (23). I believe that future experiments, both for laser-pellet interactions and heating in tokamaks, will provide further insight into the nature of the truly nonlinear processes.

The author is grateful to Dr. John Kulp for help with the computational facilities and Dr. Abhay Ram for valuable discussions. We thank Professor A. Bers and Dr. C. F. C. Karney for a critical reading of the manuscript. This work was supported by the U. S. Department of Energy and the National Science Foundation.

[1]R. E. Aamodt and M. C. Vella, Phys. Rev. Lett. 39, 1273 (1977).

[2]V. B. Krapchev, Research Laboratory of Electronics, Massachusetts Institute of Technology Report No. PRR 78/30 (unpublished).

[3]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1965).

[4]H. Motz and C. J. H. Watson, Adv. Electron. 23, 153 (1967).