Forces and Potentials in a Plasma LAPTAG plasma notes

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Plasmas are made up of charged particles and sometimes neutral atoms. A typical Laptag plasma may have a density of $n_e = n_I = 10^{11}\ \text{cm}^{-3}$. If the Laptag plasma is a cylinder 30 cm in diameter and 100 cm long its volume is $V = 100 \bullet \pi \bullet 15^2 = 225\pi \times 10^2 \simeq 7 \times 10^4\ \text{cm}^3$, and have a total number of ions/electrons of $N = 7 \times 10^{15}$. (Seven million trillion). Each particle obeys force laws and has a differential equation describing its motion in space and time. All the motions are coupled because the electric and magnetic fields are due to all the particles. They is no way to solve that many simultaneous equations so we have to use other methods. What are the force laws? From Newton we have F=ma. Force is mass times acceleration. The correct way to write this is in differential form

(1) $\vec{F} = m\vec{a} = m\dfrac{d^2\vec{r}}{dt^2}$ Force is a vector, so is acceleration and position.

There is the force of gravity but we ignore it as it is much smaller than electromagnetic forces. The force on a single particle of charge q and mass m is called the electromagnetic force law and is given by

(2) $\vec{F} = q\left(\vec{\text{v}} \times \vec{B} + \vec{E}\right)$ E and B are the electric and magnetic fields and v is the velocity; all are vectors. The first term is a velocity dependent force and the X is the vector cross product. If we write out the magnetic component (called the Lorentz force)

(3) $\vec{F} = q\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \text{v}_x & \text{v}_y & \text{v}_z \\ B_x & B_y & B_z \end{pmatrix}$ This is rectangular coordinates with the 3 axis the familiar x,y,z axis. This can be written explicitly as

(4) $$\vec{F} = q\left(\text{v}_y B_z - \text{v}_z B_y\right)\hat{i} + q\left(\text{v}_z B_x - \text{v}_x B_z\right)\hat{j} + q\left(\text{v}_x B_y - \text{v}_y B_x\right)\hat{k}$$

Here $\hat{i}, \hat{j}, \hat{k}$ are unit vectors in the x,y,and z directions. These are “pointers” with length 1. This means $\hat{i} \bullet \hat{i} = 1$ (vector dot product)

The magnetic field now gives us a “preferred direction” in space as form equation (1) there is no magnetic force on particles moving along the magnetic field. We can write the velocity as $\vec{\text{v}} = \vec{\text{v}}_{\parallel} + \vec{\text{v}}_{\perp}$. Now

(5) $\vec{F} = q\left(\vec{E} + \left(\vec{v}_{\parallel} + \vec{v}_{\perp}\right) \times \vec{B}\right) = q\left(\vec{E} + \left(\vec{v}_{\perp}\right) \times \vec{B}\right)$

Let us assume that $\vec{B} = B_0\left(-\hat{k}\right)$ and there is no electric field. Then

(6) $$\vec{F}_{\perp} = q\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \mathrm{v_x} & \mathrm{v_y} & 0 \\ 0 & 0 & \mathrm{-B_0} \end{pmatrix}$$

From the z component we get $m\dfrac{d\mathrm{v}_z}{dt} = 0$; $\mathrm{v}_z = \mathrm{v}_{z0}$. The z component of the velocity is not changed and remains whatever it was when the magnetic field appears. The other components are:

$$m\frac{d\mathrm{v}_x}{dt} = -\mathrm{v}_y B_0 \quad ; \quad m\frac{d\mathrm{v}_y}{dt} = \mathrm{v}_x B_0$$

Take the derivative of the equation for $\dfrac{d\mathrm{v}_x}{dt}$ and substitute the second equation to get

$$\frac{d^2\mathrm{v_x}}{\mathrm{dt}^2} = -\frac{qB_0}{m}\frac{d\mathrm{v_y}}{dt} = -\left(\frac{qB_0}{m}\right)^2 \mathrm{v_x}$$

Now there are two equations for the x,y velocity components:

(7) $$\frac{d^2\mathrm{v_x}}{\mathrm{dt}^2} = -\left(\frac{qB_0}{m}\right)^2 \mathrm{v_x}$$
$$\frac{d^2\mathrm{v_y}}{\mathrm{dt}^2} = -\left(\frac{qB_0}{m}\right)^2 \mathrm{v_y}$$

If we define $\omega_c = \dfrac{qB_0}{m}$ we get two equations that we have seen before. This quantity is the cyclotron frequency. The solution using the method we learned at the outset is $\mathrm{v}_x = Ae^{i(\omega_c t + \varphi)} + D$. The solution depends on boundary conditions. Let us assume that at t=0 $\vec{\mathrm{v}} = \mathrm{v}_x\hat{i}$ $(\mathrm{v}_y = 0)$. Then

(5) $\mathrm{v}_x = \mathrm{Re}(Ae^{i(\omega_c t + \varphi)}) + D = A\cos\left(\omega_c t + \varphi\right)$

A and φ are constants that depend on the initial conditions. Suppose at t=0 $\vec{\mathrm{v}} = \mathrm{v}_x$. This means that D=0. $\mathrm{v}_x = \mathrm{Re}\left\{Ae^{i\omega_c t}\right\} = A\cos\left(\omega_c t\right), \quad \varphi$=0 . To find the y component of the velocity use $m\frac{d\mathrm{v}_x}{dt} = -q\mathrm{v}_y B_0$:

$$mA\omega_c(-\sin\left(\omega_c t\right)) = -q\mathrm{v}_y B_0$$

$$\mathrm{v}_y = \frac{\omega_c m}{qB_0}\sin\left(\omega_c t\right) = A\sin\left(\omega_c t\right).$$

Here ω_c is the angular cyclotron frequency. For electrons the cyclotron frequency fc is

(7) $f_c = \frac{\omega_c}{2\pi} = 2.8\times10^6 B \;$ (B in Gauss)

(What is the difference between ω_c and f_c? Let us assume that the total velocity perpendicular to B is $\mathrm{v}_\perp$. There is no preferred direction so since $\mathrm{v} = \sqrt{\mathrm{v}_x^2 + \mathrm{v}_y^2} = \mathrm{v}_\perp$ then $A = \mathrm{v}_\perp$.

We can find the x and y positions as a function of time by integrating the velocities:

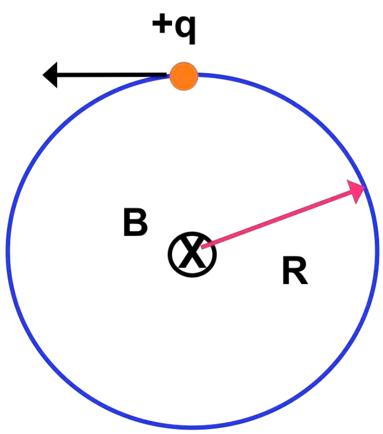
$$x - x_0 = \frac{-\mathrm{v}_0}{\omega_c}\sin\left(\omega_c t\right) \quad \text{and}$$

$$y - y_0 = \frac{\mathrm{v}_0}{\omega_c}\cos\left(\omega_c t\right)$$

There is no preferable direction perpendicular to the z axis which means the particle moves in a circle.

(8) $R = \sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2} = \frac{\mathrm{v}_0}{\omega_c} = \frac{m\mathrm{v}_0}{qB}$

R is called the cyclotron radius. The motion of a charged particle in the magnetic field is illustrated :



Positive charges spin CCW and negative ones CW if B points into the page. Note the spinning particle makes a tiny current and as you will find out next quarter this current makes a magnetic field that always opposes the background field B.

What is a plasma? It is not just a bunch of charges but it behaves collectively. The first example is Debye shielding. Let us place a sphere of positive charge inside of a plasma. What is the potential and field as a function of radius from the center of the charge. We know in vacuum the potential is $V(r)=\frac{q}{4\pi\varepsilon_0 r}$. Let us assume that the potential is given by $\vec{E}=-\nabla\phi$, that is we ignore the displacement current, a current due to rapidly changing magnetic fields.

$\nabla\bullet\vec{\mathbf{E}}=\frac{\rho}{\varepsilon_0}=-\nabla^2\phi$. In spherical coordinates: $\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$

(9) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)=-\frac{e}{\varepsilon_0}\left(n_I-n_e\right)$. This assumes that there is a local charge imbalance.

The electrons are far more mobile than the ions (at least 1860 times less massive) so this is possible.

To get to the next step we must get a model of n(ϕ) for the ions and electrons. Since there are so many particles we will use statistics. Consider the probability of find an electron in the range $\vec{r}+\Delta r,\ \vec{\mathrm{v}}+\Delta\vec{\mathrm{v}}$ at a given time t. This is given by the statistical probability function $f(\vec{r},\vec{\mathrm{v}},\mathrm{t})$. If we fix a time and spatial position the area under the curve A Gaussian for one dimension in v, v=v_x at a fixed r and t is shown in figure 1.

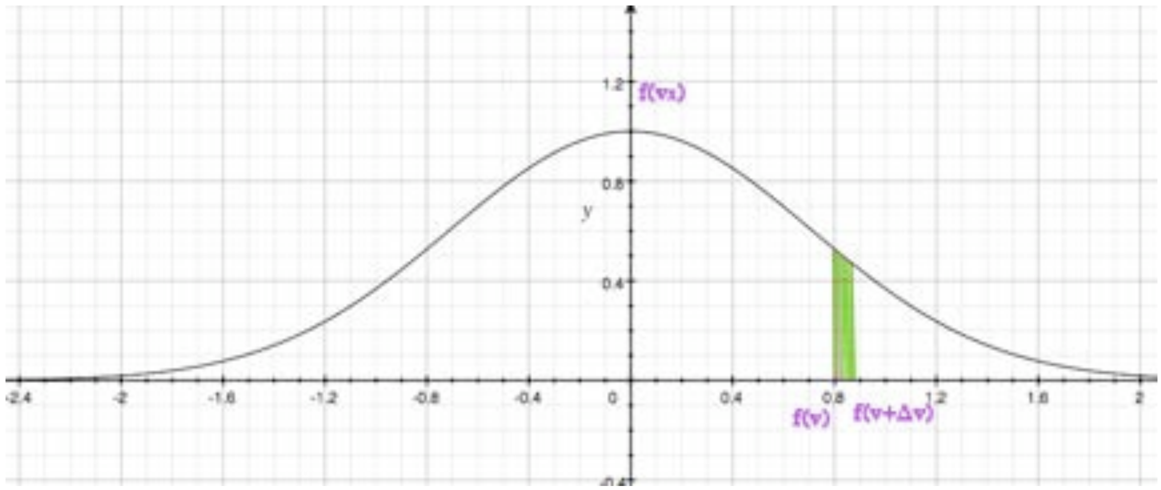


Figure 1 A Gaussian in 1-D velocity space

$\int_{v}^{v+\Delta v} d\vec{v} f\left(\vec{r}_0, \vec{v}, t_0\right)$ is the number density of particles between v and v+dv, at r_0 (area of the green stripe in figure 1).. Note there is a different f for ions and electrons and f is 7 dimensional. Next assumption: The spatial and velocity components of f are separable i.e. f = f(v)f(r,t). This means that

$f(\vec{r}, \vec{v}, t) = f_n(\vec{v}) n(\vec{r}, t)$. Note this assumes that at each location the distribution in velocity space is the same. This, of course, is not true all of the time. If we integrate over all velocity space we have the density distribution. $n(\vec{r},t)=\int_{-\infty}^{\infty} f(\vec{r}, \vec{v}, t)\, d^3\vec{v}$. The integration is over all three dimensions in velocity space. The total number of particles is $n(t)=\int_{-\infty}^{\infty} n(\vec{r},t)\, d^3\vec{r} = \int_{-\infty}^{\infty} d^3\vec{r} \int_{-\infty}^{\infty} f(\vec{r}, \vec{v}, t)\, d^3\vec{v}$.

The Maxwellian is a special case of a distribution function in thermal equilibrium. For this case $f(v) = An(\vec{r}, t) e^{\frac{mv^2}{2KT}}$, where K is Boltzman's constant (K=1.38X10^{-23} Joules/degree K), and A is a normalization $A = \left(\frac{m}{2\pi KT}\right)^{\frac{\dim s}{2}}$. Dims is the number of dimensions so for 3D the factor is 3/2. Also in spherical coordinates $d^3\vec{v} = v^2 dv \sin\theta d\theta d\phi$ and $\int \sin\theta d\theta d\phi = 4\pi$ the solid angle of the celestial sphere.

Taking moments of the distribution function gives averages (we will have more to say about this later) for example

10) $\langle \mathrm{v} \rangle = 4\pi \left(\frac{m}{2\pi KT} \right)^{\frac{3}{2}} \int_{-\infty}^{\infty} \mathrm{v} e^{\frac{m\mathrm{v}^2}{2\mathrm{KT}}} d^3\mathrm{v}$. (This is zero in this case).

The second moment is proportional to the kinetic energy. T is the temperature of the species in question and m is its mass.

11) $\langle \mathrm{v}^2 \rangle = 4\pi \left(\frac{m}{2\pi KT} \right)^{\frac{3}{2}} \int_{-\infty}^{\infty} \mathrm{v}^2 e^{\frac{m\mathrm{v}^2}{2\mathrm{KT}}} d^3\mathrm{v}$. (which is non-zero).

Now let us assume the ions are cold and the electrons are not. If the ions are cold then v=0 and there is no velocity dependence. Now let assume that the ions are distributed uniformly throughout all of space. If they are stone cold they are not moving randomly and we also assume they are not drifting in unison one way or the other. Let the uniform ion density be n_0. The next assumption is that there are no neutral particles, the plasma is fully ionized.

As for the electrons we have to add something to the distribution function, which reflects that they will be repelled from regions with a large negative potential. Only electrons with kinetic energy large than the negative potential could be found at that location. We modify the distribution function according to:

12) $f_e(r,\mathrm{v},\mathrm{t}) = e^{\frac{e\phi(r)}{kT_e}} \int_{-\infty}^{\infty} \left(\frac{m}{2\pi KT} \right)^{\frac{3}{2}} n(t) e^{-\frac{m\mathrm{v}^2}{2\mathrm{KT_e}}} d^3\mathrm{v} = n_0 e^{\frac{e\phi(r)}{kT_e}}$ (we assume no temporal dependence and integrate over velocity space) Finally equation 9) becomes:

(13) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r} \right) = \frac{en_0}{\varepsilon_0}\left(e^{\frac{e\varphi}{KT_e}} - 1 \right)$

This equation is nonlinear and has no closed analytical solution. Let us assume that $e\phi << KT_e$. The potential energy associated with our charge perturbation is less than the average electron energy. We can expand the exponential term : $e^{\frac{e\phi}{kT_e}} \cong 1 + \frac{e\phi}{kT_e} + \frac{1}{2}\left(\frac{e\phi}{kT_e} \right)^2 + + + higher \text{ order terms}$. Then the equation becomes:

14) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r} \right) = \frac{e^2 n_0 \phi}{\varepsilon_0 KT_e}$. The solution is

15) $\phi(r) = \frac{De^{-\frac{r}{\lambda_D}}}{r}$ with $\lambda_\mathrm{D} = \sqrt{\frac{\varepsilon_0 KT_e}{ne^2}} = 7.4X10^2\sqrt{\frac{T_e}{n}}$ (cm)

The electron temperature is expressed in electron volts in the numerical expression on the right. One electron volt corresponds to 11,600° K. If the ions were hot the temperature would have an ion contribution.

The difference between a 1/r and the Debye length drop off is shown in figure 2

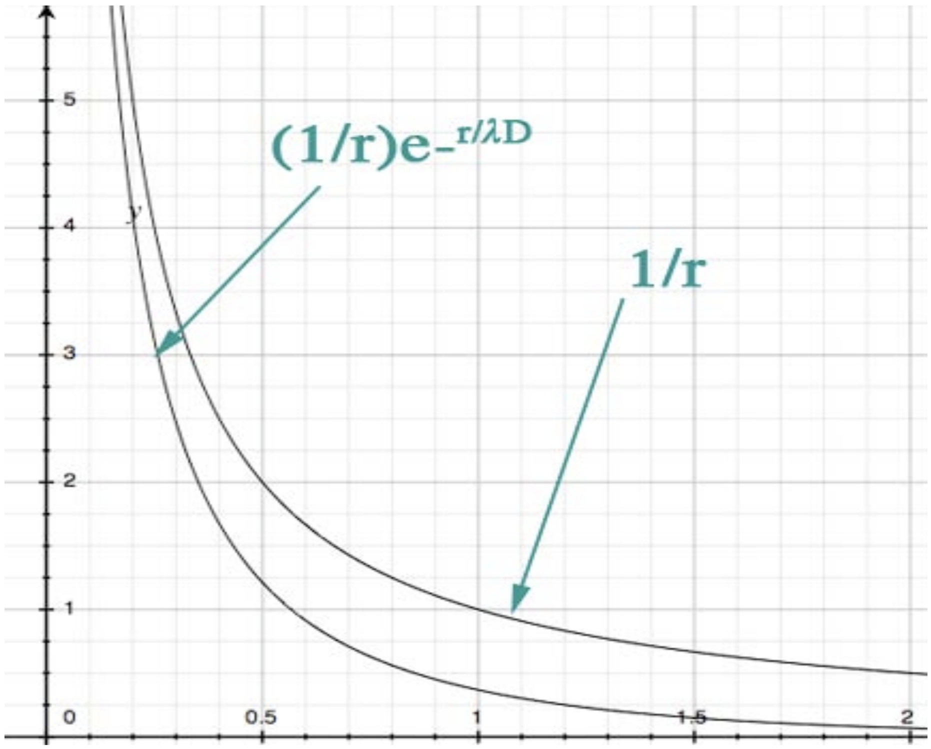


Figure 2 Drop off Debye versus 1/r

Let's revisit all of the assumptions we made

1) The ions are motionless and distributed uniformly.
2) The electrons are distributed according to Maxwell-Boltzman statistics and the distribution function is Gaussian
3) There are no neutral particles. We don't have to worry about collisions of the electrons with neutrals
4) $e\phi << KT_e$ and we can linearize equation 1a.
5) The spatial and velocity components of the distribution function are separable.

The Debye length can be vastly different depending upon the plasma we are studying Figure 3 shows the wide variety of Debye lengths in plasmas which range from cold and tenuous to hot and dense

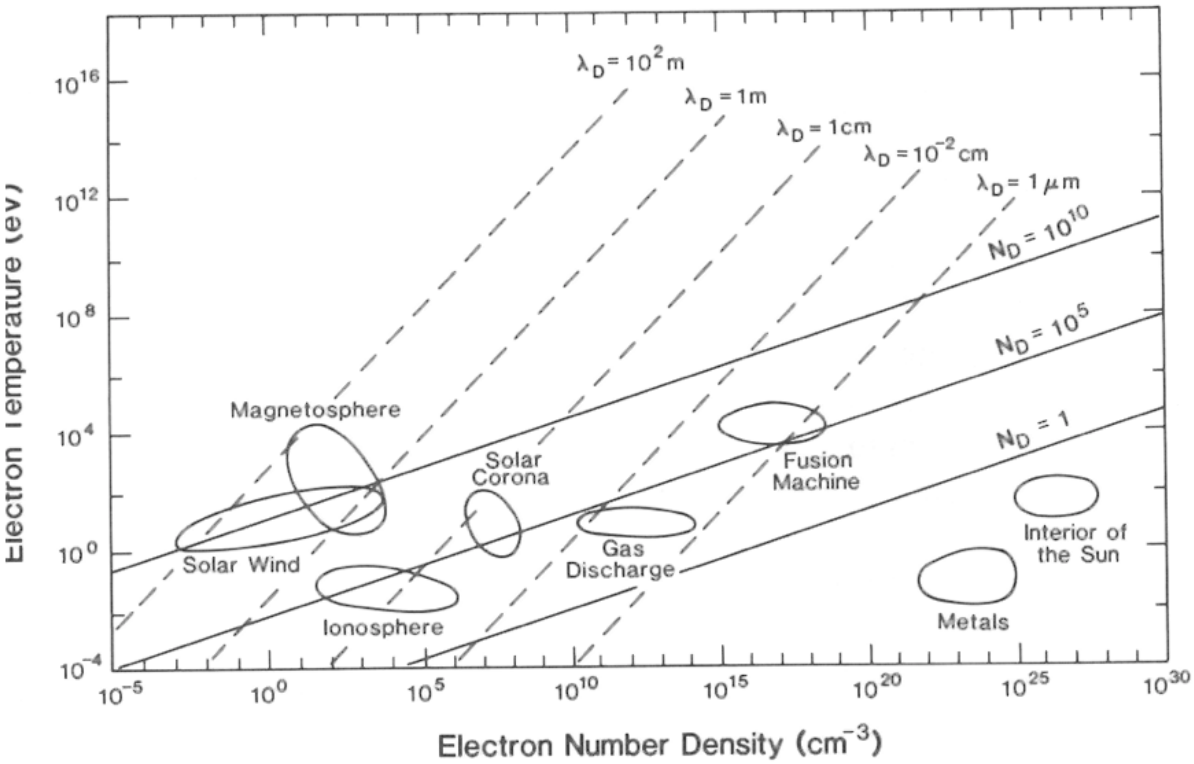


Figure 4. Ranges of Debye lengths. In the Laptag plasma ("gas discharge") it is of order 0.01 cm and can be 100 meters to many miles in space.

There is a set of equations called Maxwell's equations that relate electric and magnetic fields to currents. We write them below as differential equations (differential forms). We will explain what they mean and use them when we discuss plasma phenomena such as waves and resonance cones. For completeness here they are:

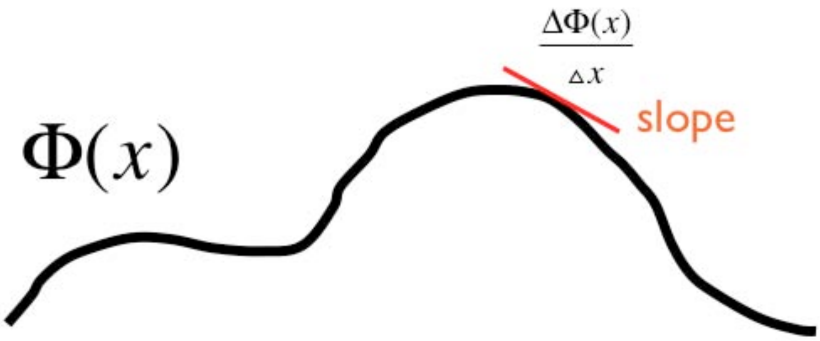
$$I)\quad \nabla \bullet \vec{\mathbf{B}} = 0 \qquad \text{II)}\quad \nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{j}} + \mu_0 \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$III)\quad \nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \text{IV)}\quad \nabla \bullet \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$$

Before we stated that the $\vec{E} = -\nabla\phi$, the electric field is the gradient of a potential. What do we mean. The electrical potential is really a potential energy, just like the gravitational potential energy. If you raise a mass in a gravitational field you increase its potential energy. It takes work on your part to do this. You first lean that work is force times distance but the correct mathematical expression is

(16) $W_{ab} = -\int_a^b \vec{F} \bullet d\vec{l}$ If a is the initial position of a mass (suppose it is on the ground so a=0) and you raise it to a height b you will do work. The force of gravity points down

$\vec{F} = mg\left(-\hat{j}\right)$ and you are moving it up $d\vec{l} = dy\hat{j}$. Therefore the work is mg(b-a) = mgb if a is on the ground (try it) and is positive. Suppose the potential changes only in the x direction $\Phi = \Phi(x)$. Then $-\nabla\Phi = -\frac{\partial\Phi}{\partial x}\hat{i} \rightarrow -\frac{\Delta\Phi}{\Delta x}(Lim\ \Delta\text{x} \rightarrow 0)$. This is simply the derivative therefore the force is the negative of the slope on a potential energy diagram.



In the figure above at the point where we draw the line the slope is negative therefore the force is positive. Let us suppose that the line is tangent to curve at point a. The force is positive at point a which means that if dx>0 you allow a marble at point a to roll downhill and it will do positive work for you. This is like dropping a ball. It can do positive work like breaking a peanut when it hits the ground.